Do Dark Pools Harm Price Discovery?*

Haoxiang Zhu

MIT Sloan School of Management

July 2012

Abstract

Dark pools are equity trading systems that do not publicly display orders. Orders in dark pools are matched within the exchange bid-ask spread without a guarantee of execution. Informed traders are more likely to cluster on the heavy side of the market and therefore face a lower execution probability in the dark pool, relative to uninformed traders. Consequently, exchanges are more attractive to informed traders, whereas dark pools are more attractive to uninformed traders. Under natural conditions, adding a dark pool alongside an exchange concentrates price-relevant information into the exchange and improves price discovery. Dark pools that operate as nondisplayed limit order books are more attractive to informed traders than dark pools that execute orders at the exchange midpoint.

Keywords: dark pools, price discovery, liquidity, fragmentation, equity market structure

JEL Classifications: G12, G14, G18

*First version: November 2010. For helpful comments, I am very grateful to Darrell Duffie, Sal Arnuk, Jonathan Berk, John Beshears, Bradyn Breon-Drish, Robert Burns, Peter DeMarzo, Thomas George, Steven Grenadier, Frank Hatheway, Dirk Jenter, Ron Kaniel, Arthur Korteweg, Ilan Kremer, Charles Lee, Han Lee, Ian Martin, Jim McLoughlin, Albert Menkveld, Stefan Nagel, Francisco Pérez-González, Paul Pfleiderer, Monika Piazzesi, Michael Ostrovsky, Martin Schneider, Ken Singleton, Jeffrey Smith, Ilya Strebulaev, Ingrid Werner (discussant), Mao Ye (discussant), Ruiling Zeng, and Jeff Zwiebel, as well as seminar participants at Stanford University, Chicago Booth, Princeton University, University of Illinois, MIT Sloan, NYU Stern, Wharton, UT Austin McCombs, Berkeley Haas, UCLA Anderson, Northwestern Kellogg, the Western Finance Association annual meeting, the NBER Market Design Working Group meeting, and the SFS Finance Cavalcade. All errors are my own. Corresponding address: MIT Sloan School of Management, 100 Main Street E62-623, Cambridge, MA 02142. E-mail: zhuh@mit.edu. Paper URL: http://ssrn.com/abstract=1712173.

1 Introduction

Dark pools are equity trading systems that do not publicly display orders. Some dark pools passively match buyers and sellers at exchange prices, such as the midpoint of the exchange bid and offer. Other dark pools essentially operate as nondisplayed limit order books that execute orders by price and time priority.

In this paper, I investigate the impact of dark pools on price discovery. Contrary to misgivings expressed by some regulators and market participants, I find that under natural conditions, adding a dark pool improves price discovery on the exchange.

According to the Securities and Exchange Commission (SEC; 2010), as of September 2009, 32 dark pools in the United States accounted for 7.9% of total equity trading volume. As of mid-2011, industry estimates from the Tabb Group, a consultancy, and Rosenblatt Securities, a broker, attribute about 12% of U.S. equity trading volume to dark pools. The market shares of dark pools in Europe, Canada, and Asia are smaller but quickly growing (International Organization of Securities Commissions, 2010).

Dark pools have raised regulatory concerns in that they may harm price discovery. The European Commission (2010), for example, remarks that "[a]n increased use of dark pools . . . raise[s] regulatory concerns as it may ultimately affect the quality of the price discovery mechanism on the 'lit' markets." The International Organization of Securities Commissions (2011) similarly worries that "the development of dark pools and use of dark orders could inhibit price discovery if orders that otherwise might have been publicly displayed become dark." According to a recent survey conducted by the CFA Institute (2009), 71% of respondents believe that the operations of dark pools are "somewhat" or "very" problematic for price discovery. The Securities and Exchange Commission (2010), too, considers "the effect of undisplayed liquidity on public price discovery" an important regulatory question. Speaking of nondisplayed liquidity, SEC Commissioner Elisse Walter commented that "[t]here could be some truth to the criticism that every share that is crossed in the dark is a share that doesn't assist the market in determining an accurate price."[1]

My inquiry into dark pools builds on a simple model of strategic venue selection by informed and liquidity traders. Informed traders hope to profit from proprietary information regarding the value of the traded asset, whereas liquidity traders wish to meet their idiosyncratic liquidity needs. Both types of traders optimally choose between an exchange and a dark pool. The exchange displays a bid and an ask and executes all submitted orders at the bid or the ask. The dark pool free-rides on exchange prices

[1] "Speech by SEC Commissioner: Opening Remarks Regarding Dark Pools," October 21, 2009.

by matching orders within the exchange's bid and ask. Unlike the exchange, the dark pool has no market makers through which to absorb excess order flow and thus cannot guarantee execution. Sending an order to the dark pool therefore involves a trade-off between potential price improvement and the risk of no execution.

Execution risk in the dark pool drives my results. Because matching in the dark pool depends on the availability of counterparties, some orders on the "heavier" side of the market—the side with more orders—will fail to be executed. These unexecuted orders may suffer costly delays. Because informed orders are positively correlated with the value of the asset and therefore with each other, informed orders are more likely to cluster on the heavy side of the market and suffer lower execution probabilities in the dark pool. By contrast, liquidity orders are less correlated with each other and less likely to cluster on the heavy side of the market; thus, liquidity orders have higher execution probabilities in the dark pool. This difference in execution risk pushes relatively more informed traders into the exchange and relatively more uninformed traders into the dark pool. Under natural conditions, this self selection lowers the noisiness of demand and supply on the exchange and improves price discovery.

The main intuition underlying my results does not hinge on the specific trading mechanisms used by a dark pool. For example, a dark pool may execute orders at the midpoint of the exchange bid and ask or operate as a nondisplayed limit order book. As I show, with both of these mechanisms, traders face a trade-off between potential price improvement and execution risk. Dark pools that operate as limit order books are, however, relatively more attractive to informed traders because limit orders can be used to gain execution priority and thus reduce execution risk. This result suggests that informed traders have even stronger incentives to trade on the exchange under a "trade-at" rule, which requires that trading venues that do not quote the best price either to route incoming orders to venues quoting the best price or to provide incoming orders with a sufficiently large price improvement over the prevailing best price. The impact of a trade-at rule on price discovery complements previous fairness-motivated arguments that displayed orders—which contribute to pre-trade transparency—should have strictly higher priority than do nondisplayed orders at the same price.[2]

Dark pools do not always improve price discovery. For example, in the unlikely event that liquidity traders push the net order flow far opposite of the informed traders, the

[2]For example, the Joint CFTC-SEC Advisory Committee (2011) has noted: "Under current Regulation NMS routing rules, venues cannot 'trade through' a better price displayed on another market. Rather than route the order to the better price, however, a venue can retain and execute the order by matching the current best price even if it has not displayed a publicly accessible quote order at that price. While such a routing regime provides order execution at the current best displayed price, it does so at the expense of the limit order posting a best price which need not receive execution."

presence of a dark pool can exacerbate the misleading inference regarding the asset value. Moreover, better price discovery needs not coincide with higher liquidity or welfare. Indeed, more informative orders often lead to better price discovery but also tend to worsen adverse selection on the exchange, which results in wider spreads and higher price impacts. The welfare implications of dark pools could naturally depend on elements outside the setting of my model, such as how price discovery and liquidity affect production decisions, asset allocation, and capital formation.[3] In addition, for analytical tractability I have abstracted from some of the trading practices that are applied in dark pools, such as "pinging," order routing, and "indication of interest" (IOI).[4] These and other procedures used by some dark pools may well contribute to concerns regarding their impact on price discovery, although these practices are distinct from the implications of execution risk, which I focus on in this paper. Finally, the price-discovery effect of dark pools complements their "size discovery" function, by which large institutional orders are executed without being revealed to the broad market. This size-discovery benefit of dark trading has been widely acknowledged by market participants and regulators, and today only a handful of dark pools execute large orders (Securities and Exchange Commission, 2010; Ready, 2012).

To the best of my knowledge, this paper is the first to show that the addition of a dark pool can improve price discovery. My finding stands in contrast to that of Ye (2011), who studies the venue choice of a large informed trader in the Kyle (1985) framework and concludes that the addition of a dark pool harms price discovery on the exchange. Ye (2011), however, assumes *exogenous* choices of trading venues by liquidity traders, whereas the *endogenous* venue choices of liquidity traders are critical to my results. Most other existing models of dark pools either exogenously fix the strategies of informed traders, as in Hendershott and Mendelson (2000), or do not consider the role of asymmetric information regarding the asset value, as in Degryse, Van Achter, and Wuyts (2009) and Buti, Rindi, and Werner (2011b). Going beyond the midpoint-matching mechanism, my study additionally reveals that dark pools with more discretion in execution prices are more attractive to informed traders.

The focus of this paper—i.e., on the fragmentation of order flow between an ex-

[3]For example, see Bond, Edmans, and Goldstein (2012) for a survey on the literature that studies the effects of financial markets on the real economy.

[4]"Pinging" orders are marketable orders that seek to interact with displayed or nondisplayed liquidity. Pinging is sometimes used to learn about the presence of large hidden orders. Order routing means sending orders from venue to venue, typically by algorithms. For example, if a dark pool cannot execute an order because there is no counterparty, the dark pool can route the order to another dark pool, which may further route the order into the market. An IOI is an electronic message that contains selected information (such as the ticker) about an order and is sent by a trading venue (such as a dark pool or a broker) to a selected group of market participants in order to facilitate a match.

change and a dark pool—differs from the focus of prior studies on competition among multiple markets. In exchange markets, for example, informed traders and liquidity traders tend to cluster by time (Admati and Pfleiderer, 1988) or by location (Pagano, 1989; Chowdhry and Nanda, 1991). However, as modeled here, informed traders cluster less with liquidity traders in the dark pool than on the exchange because informed traders face higher execution risk in the dark pool. Related to the effect captured by my model, Easley, Keifer, and O'Hara (1996) suggest that the purchase of retail order flows ("cream-skimming") by regional exchanges results in higher order informativeness on the NYSE. In contrast with the mechanism studied in their paper, in my model dark pools rely on self selection, rather than intermediaries, to separate, at least partially, informed traders from liquidity traders.

My results have several empirical implications. For example, the model predicts that higher order imbalances tend to cause lower dark pool activity; higher volumes of dark trading lead to wider spreads and higher price impacts on exchanges; volume correlation across stocks is higher on exchanges than in dark pools; and informed traders more actively participate in dark pools when asymmetric information is more severe or when the dark pool allows more discretion in execution prices. Section 6 discusses these implications, as well as discussing recent, related empirical evidence documented by Ready (2012), Buti, Rindi, and Werner (2011a), Ye (2010), Nimalendran and Ray (2012), Degryse, de Jong, and van Kervel (2011), Jiang, McInish, and Upson (2011), O'Hara and Ye (2011), and Weaver (2011), among others.

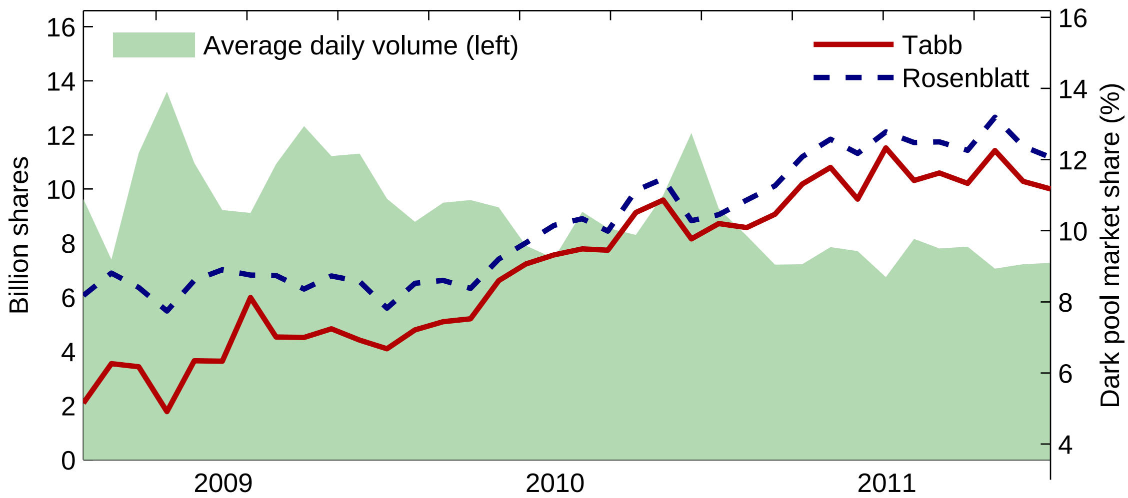
2 An Overview of Dark Pools

This section provides an overview of dark pools. I discuss why dark pools exist, how they operate, and what distinguishes them from each other. For concreteness, I tailor this discussion for the market structure and regulatory framework in the United States. Dark pools in Europe, Canada, and Asia operate similarly.

Before 2005, dark pools had low market share. Early dark pools were primarily used by institutions to trade large blocks of shares without revealing their intentions to the broad market, in order to avoid being front-run.[5] A watershed event for the U.S. equity market was the adoption in 2005 of Regulation National Market System, or Reg NMS (Securities and Exchange Commission, 2005), which abolished rules that had protected the manual quotation systems of incumbent exchanges. In doing so,

[5]Such predatory trading is modeled by Brunnermeier and Pedersen (2005) and Carlin, Lobo, and Viswanathan (2007).

Figure 1: U.S. equity trading volume and the market share of dark pools. The left axis plots the daily consolidated equity trading volume in the United States, estimated by Tabb Group. The right axis plots the market shares of dark pools as a percentage of the total consolidated volume, estimated by Tabb Group and Rosenblatt Securities.



Reg NMS encouraged newer and faster electronic trading centers to compete with the incumbents. Since Reg NMS came into effect, a wide variety of trading centers have been established. As of September 2009, the United States had about 10 exchanges, 5 electronic communication networks (ECNs), 32 dark pools, and over 200 broker-dealers (Securities and Exchange Commission, 2010). Exchanges and ECNs are referred to as transparent, or "lit," venues; dark pools and broker-dealer internalization are considered opaque, or "dark," venues. In Europe, the adoption in 2007 of the Markets in Financial Instruments Directive (MiFID) similarly led to increased competition and a fast expansion of equity trading centers.[6]

Figure 1 shows the consolidated volume of U.S. equity markets from July 2008 to June 2011, as well as the market share of dark pools during the same periods, estimated by Tabb Group and Rosenblatt Securities. According to their data, the market share of dark pools roughly doubled from about 6.5% in 2008 to about 12% in 2011, whereas consolidated equity volume dropped persistently from about 10 billion shares per day in 2008 to about 7 billion shares per day in 2011. A notable exception to the decline in consolidated volume occurred around the "Flash Crash" of May 2010.

Dark pools have gained market share for reasons that go beyond recent regulations

[6]For example, according to CFA Institute (2009), European equity market had 92 regulated markets (exchanges), 129 "multilateral trading facilities" (MTFs), and 13 "systematic internalizers" as of Septempber 2010. For more discussion of MiFID and European equity market structure, see European Commission (2010).

designed to encourage competition. Certain investors, such as institutions, simply need nondisplayed venues to trade large blocks of shares without alarming the broad market. This need has increased in recent years as the order sizes and depths on exchanges have declined dramatically (Chordia, Roll, and Subrahmanyam, 2011). Further, dark pools attract investors by offering potential price improvements relative to the best prevailing bid and offer on exchanges. Finally, broker-dealers handling customer orders have strong incentives to set up their own dark pools, where they can better match customer orders internally and therefore save trading fees that would otherwise be paid to exchanges and other trading centers.

Dark pools differ from each other in many ways. We can categorize them, roughly, into the three groups shown in Table 1.

Dark pools in the first group match customer orders by acting as agents (as opposed to trading on their own accounts). In this group, transaction prices are typically derived from lit venues. These derived prices include the midpoint of the national best bid and offer (NBBO) and the volume-weighted average price (VWAP). Examples in this group include block-crossing dark pools such as ITG Posit and Liquidnet.[7] Posit crosses orders a few times a day at scheduled clock times (up to some randomization), although in recent years it has also offered continuous crossing. Liquidnet is integrated into the order-management systems of institutional investors and alerts potential counterparties when a match is found. Instinet is another agency broker that operates scheduled and continuous dark pools. Dark pools operated by exchanges typically use midpoint matching as well. Because Group-1 dark pools rely on lit venues to determine execution prices, they typically do not provide direct price discovery.

Within the second group, dark pools operate as continuous nondisplayed limit order books, accepting market, limit, or "pegged" orders.[8] This group includes many of the dark pools owned by major broker-dealers, including Credit Suisse Crossfinder, Goldman Sachs Sigma X, Citi Match, Barclays LX, Morgan Stanley MS Pool, and UBS PIN. Unlike Group-1 dark pools that execute orders at the market midpoint or VWAP, Group-2 dark pools derive their own execution prices from the limit prices of submitted orders. Price discovery can therefore take place. Another difference is that Group-2 dark pools may contain proprietary order flows from the broker-dealers that operate them. In this sense, these dark pools are not necessarily "agency only."

Dark pools in the third group act like fast electronic market makers that immediately accept or reject incoming orders. Examples include Getco and Knight. Like the second

[7] See also Ready (2012) for a discussion of these two dark pools.

[8] Pegged orders are limit orders with the limit price set relative to an observable market price, such as the bid, the offer, or the midpoint. As the market moves, the limit price of a pegged order moves accordingly.

Table 1: Dark pool classification by trading mechanisms.

Types	Examples	Typical features
Matching at exchange prices	ITG Posit, Liquidnet, Instinet	Mostly owned by agency brokers and exchanges; typically execute orders at midpoint or VWAP, and customer-to-customer
Nondisplayed limit order books	Credit Suisse Crossfinder, Goldman Sachs Sigma X, Citi Match, Barclays LX, Morgan Stanley MS Pool, UBS PIN	Most broker-dealer dark pools; may offer some price discovery and contain proprietary order flow
Electronic market makers	Getco and Knight	High-speed systems handling immediate-or-cancel orders; typically trade as principal

group, transaction prices on these platforms are not necessarily calculated from the national best bid and offer using a transparent rule. In contrast with dark pools in Groups 1 and 2, Group-3 dark pools typically trade on their own accounts as principals (as opposed to agents or marketplaces).

Appendix A discusses additional institutional features of dark pools. Overviews of dark pools and nondisplayed liquidity are also provided by Johnson (2010), Butler (2007), Carrie (2008), Securities and Exchange Commission (2010), European Commission (2010), CSA/IIROC (2009), and International Organization of Securities Commissions (2011).

3 Modeling the Exchange and the Dark Pool

This section presents a two-period model of trading-venue selection. Each trader chooses whether to trade on a transparent exchange or in a dark pool. The dark pool modeled in this section passively matches orders at the midpoint of the exchange's bid and ask. Section 4 models a dark pool that operates as a nondisplayed limit order book. The order-book setting provides additional insights regarding the effect of the dark pool crossing mechanism for price discovery. A dynamic equilibrium with sequential arrival of traders is characterized in Section 5. A glossary of key model variables can be found in Appendix D.

3.1 Markets and traders

There are two trading periods, denoted by $t = 1, 2$. At the end of period 2, an asset pays an uncertain dividend v that is equally likely to be $+\sigma$ or $-\sigma$. Thus, $\sigma > 0$ is the volatility of the asset value. The asset value v is publicly revealed at the beginning of period 2.

Two trading venues operate in parallel: a lit exchange and a dark pool. The exchange is open in periods 1 and 2. On the exchange, a risk-neutral market maker sets competitive bid and ask prices. Market orders sent to the exchange arrive simultaneously. Exchange buy orders are executed at the ask; exchange sell orders are executed at the bid. The exchange here is thus similar to that modeled by Glosten and Milgrom (1985).[9] After period-1 orders are executed, the market maker announces the volume V_b of exchange buy orders and the volume V_s of exchange sell orders. The market maker also announces the exchange closing price P_1, which is the expected asset value, conditional on V_b and V_s. The closing price P_1 is also the price at which the market maker is willing to execute a marginal order at the end of period 1. A key objective of this section is to analyze price discovery, that is, the informativeness of these announcements, in particular P_1, for the fundamental value v of the asset.

The dark pool executes (or "crosses") orders in period 1 and is closed in period 2. Closing the the dark pool in period 2 is without loss of generality because once the dividend v is announced in period 2, exchange trading is costless. An order submitted to the dark pool is not observable to anyone but the order submitter. The execution price of dark pool trades is the midpoint of the exchange bid and ask, also known simply as the "midpoint" or "mid-market" price. In the dark pool, orders on the "heavier side"—the buyers' side if buy orders exceed sell orders, and the sellers' side if sell orders exceed buy orders—are randomly selected for matching with those on the "lighter" side. For example, if the dark pool receives Q_B buy orders and $Q_S < Q_B$ sell orders, all of the same size, then Q_S of the Q_B buy orders are randomly selected, equally likely, to be executed against the Q_S sell orders at the mid-market price. Unmatched orders are returned to the order submitter at the end of period 1. As described in Section 2, this midpoint execution method is common in dark pools operated by agency brokers and exchanges. An alternative dark pool mechanism, a nondisplayed limit order book, is modeled in Section 4.

For-profit traders and liquidity traders, all risk-neutral, arrive at the beginning of period 1. There is an infinite set of infinitesimal traders of each type. For-profit traders

[9] As I describe shortly, the model of this section is not exactly the same as that of Glosten and Milgrom (1985) because orders here arrive in batches, instead of sequentially. Sequential arrival of orders is considered in Section 5.

have a mass of $\bar{\mu} > 0$, and can potentially trade one unit of the asset per capita.[10] For-profit traders can acquire, at a cost, perfect information about v, and thus become informed traders. These information-acquisition costs are distributed across for-profit traders, with the cumulative distribution function $F : [0, \infty) \to [0, 1]$. After observing v, informed traders submit buy orders (in either venue) if $v = +\sigma$ and submit sell orders if $v = -\sigma$. For-profit traders who do not acquire the information do not trade. I let μ_I be the mass of informed traders; their signed trading interest is therefore $Y = \text{sign}(v) \cdot \mu_I$.

Liquidity buyers and liquidity sellers arrive at the market separately (not as netted). The mass Z^+ of liquidity buy orders and the mass Z^- of liquidity sell orders are non-negative, independent and identically distributed on $[0, \infty)$ with positive density functions, and infinitely divisible. Infinite divisibility means that, for each integer n, the total liquidity buy orders Z^+ can be viewed as the aggregate demand by n liquidity buyers, whose order sizes are independently and identically distributed random variables. A similar construction applies for the total liquidity sell orders Z^-. Thus, we can interpret a market with infinitely many liquidity traders as the "limiting case" of a market with n liquidity buyers and n liquidity sellers as $n \to \infty$.[11] In particular, because, in the limit, each liquidity trader's order size has zero mean and zero variance, the conditional joint distribution of Z^+ and Z^-, given this liquidity trader's order size, is the same as the unconditional joint distribution of Z^+ and Z^-.[12] I denote by $0.5\mu_z$

[10] Trading one unit per capita is without loss of generality because each informed trader is infinitesimal and has zero mass. As long as per-capita trading size is finite, an informed trader's order still has zero mass, and the qualitative nature of equilibria does not change.

[11] More specifically, for each integer n, Z^+ can be constructed as the sum of n independently and identically distributed random variables $\{Z^+_{in}\}$. That is,

$$Z^+ \sim \sum_{i=1}^{n} Z^+_{in}.$$

Note that the distribution of Z^+_{in} depends on n. I assume that the variance of Z^+_{in} is finite. Similarly, there exist n i.i.d. random variables Z^-_{in} such that

$$Z^- \sim \sum_{i=1}^{n} Z^-_{in}.$$

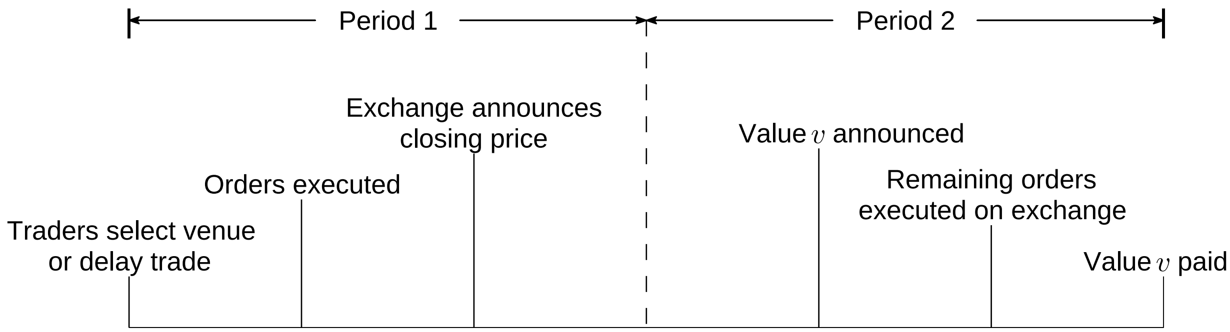
In this setting, $\{Z^+_{in}\}$ and $\{Z^-_{in}\}$ can be viewed as the order sizes of n liquidity buyers and n liquidity sellers. As $n \to \infty$, the mean and variance of Z^+_{in} and Z^-_{in} converge to zero, and liquidity buyers and sellers become infinitesimal.

[12] We denote by Φ the probability distribution of Z^+ and show that, for each i, the conditional distribution of $\sum_{j=1}^{n} Z^+_{jn}$, given Z^+_{in}, converges to the prior distribution of Z^+ as $n \to \infty$. That is, for all $z > 0$, $\Phi(z \mid Z^+_{in}) \to \Phi(z)$ as $n \to \infty$. By the independence of Z^+_{in} and $\{Z^+_{jn}\}_{j \neq i}$, this amounts to showing that Z^+_{in} converges to zero in distribution. Indeed, for any $z > 0$, using Markov's inequality and the fact that $\mathbb{E}(Z^+_{in})$ converges to zero as $n \to \infty$, we have

$$\mathbb{P}(Z^+_{in} < z) = 1 - \mathbb{P}(Z^+_{in} \geq z) \geq 1 - \frac{\mathbb{E}(Z^+_{in})}{z} \to 1, \text{as } n \to \infty.$$

Similarly, the conditional distribution of Z^-, given Z^+_{in}, converges in n to the prior distribution of Z^-. The proof

Figure 2: Time line of the two-period model.



the mean of Z^+ (and Z^-) and by $0.5\sigma_z^2$ the variance of Z^+ (and Z^-).

Liquidity traders must hold collateral to support their undesired risky positions. For each liquidity trader, the minimum collateral requirement per unit asset held is the expected loss, conditional on a loss, of her undesired position. For example, a liquidity buyer who is already short one unit of the asset has a loss of σ if $v = \sigma$, and a gain of σ if $v = -\sigma$. The collateral requirement in this case is σ. For trader i, each unit of collateral has a funding cost of γ_i per period. A delay in trade is therefore costly. These funding costs $\{\gamma_i\}$ are distributed across liquidity traders, with a twice-differentiable cumulative distribution function $G : [0, \Gamma) \to [0, 1]$, for some $\Gamma \in (1, \infty]$. Failing to trade in period 1, liquidity buyer i thus incurs a delay cost of

$$c_i = \gamma_i \mathbb{E}[\max(v, 0) \,|\, v > 0] = \gamma_i \sigma \tag{1}$$

per unit of undesirable asset position. A like delay cost applies to liquidity sellers. We could alternatively interpret this delay cost as stemming from risk aversion or illiquidity. The key is that liquidity traders differ in their desires for immediacy, captured by the delay cost $c_i = \gamma_i \sigma$. The delay costs of informed traders, by contrast, stem from the loss of profitable trading opportunities after v is revealed in period 2.

Finally, random variables v, Z^+, Z^-, and the costs of information-acquisition and delay are all independent, and their probability distributions are common knowledge. Realizations of Y, Z^+ and Z^- are unobservable, with the exception that informed traders observe v, and hence know Y. Informed and liquidity traders cannot post limit orders on the exchange; they can trade only with the exchange market maker or by sending orders to the dark pool.

Figure 2 illustrates the sequence of actions in the two-period model.

for a liquidity seller's inference is symmetric.

3.2 Equilibrium

An equilibrium consists of the quoting strategy of the exchange market maker, the market participation strategies of for-profit traders, and the trading strategies of informed and liquidity traders. In equilibrium, the competitive market maker breaks even in expectation, and all traders maximize their expected net profits.

Specifically, I let α_e and α_d be candidates for the equilibrium fractions of liquidity traders who, in period 1, send orders to the exchange and to the dark pool, respectively. The remainder, $\alpha_0 = 1 - \alpha_e - \alpha_d$, choose not to submit orders in period 1 and delay trade to period 2. We let β be the period-1 fraction of informed traders who send orders to the dark pool. The remaining fraction $1 - \beta$ of informed traders trade on the exchange. (Obviously, informed traders never delay their trades as they will have lost their informational advantage by period 2.) Once the asset value v is revealed in period 2, all traders who have not traded in period 1—including those who deferred trading and those who failed to execute their orders in the dark pool—trade with the market maker at the unique period-2 equilibrium price of v.

I first derive the equilibrium exchange bid and ask, assuming equilibrium participation fractions $(\beta, \alpha_d, \alpha_e)$. Because of symmetry and the fact that the unconditional mean of v is zero, the midpoint of the market maker's bid and ask is zero. Therefore, the exchange ask is some $S > 0$, and the exchange bid is $-S$, where S is the exchange's effective spread, the absolute difference between the exchange transaction price and the midpoint. For simplicity, I refer to S as the "exchange spread." As in Glosten and Milgrom (1985), the exchange bid and ask are set before exchange orders arrive. Given the participation fractions $(\beta, \alpha_d, \alpha_e)$, the mass of informed traders on the exchange is $(1 - \beta)\mu_I$, and the expected mass of liquidity traders on the exchange is $\alpha_e \mathbb{E}(Z^+ + Z^-) = \alpha_e \mu_z$. Because the market maker breaks even in expectation, we have that

$$0 = -(1 - \beta)\mu_I(\sigma - S) + \alpha_e \mu_z S, \tag{2}$$

which implies that

$$S = \frac{(1 - \beta)\mu_I}{(1 - \beta)\mu_I + \alpha_e \mu_z}\, \sigma. \tag{3}$$

The dark pool crosses orders at the mid-market price of zero.

Next, I derive the equilibrium mass μ_I of informed traders. Given the value σ of information and the exchange spread S, the net profit of an informed trader is $\sigma - S$. The information-acquisition cost of the marginal for-profit trader, who is indifferent between paying for information or not, is also $\sigma - S$. Because all for-profit traders with lower information-acquisition costs become informed, the mass of informed traders in equilibrium is $\bar{\mu}F(\sigma - S)$, by the exact law of large numbers (Sun, 2006). We thus have

$$\mu_I = \bar{\mu}F\left(\sigma - S\right) = \bar{\mu}F\left(\frac{\alpha_e \mu_z}{(1-\beta)\mu_I + \alpha_e \mu_z}\,\sigma\right). \tag{4}$$

For any fixed $\beta \geq 0$ and $\alpha_e > 0$, (4) has a unique solution $\mu_I \in (0, \bar{\mu})$.

Finally, I turn to the equilibrium trading strategies. Without loss of generality, I focus on the strategies of buyers. In the main solution step, I calculate the expected payoffs of an informed buyer and a liquidity buyer, on the exchange and in the dark pool. The equilibrium is then naturally determined by conditions characterizing marginal traders who are indifferent between trading on the exchange and in the dark pool.

Suppose that $\alpha_d > 0$. Because informed buyers trade in the same direction, they have the dark pool crossing probability of

$$r^- = \mathbb{E}\left[\min\left(1, \frac{\alpha_d Z^-}{\alpha_d Z^+ + \beta\mu_I}\right)\right], \tag{5}$$

where the denominator and the numerator in the fraction above are the masses of buyers and sellers in the dark pool, respectively. Liquidity buyers, on the other hand, do not observe v. If informed traders are buyers, then liquidity buyers have the crossing probability r^- in the dark pool. If, however, informed traders are sellers, then liquidity buyers have the crossing probability

$$r^+ = \mathbb{E}\left[\min\left(1, \frac{\alpha_d Z^- + \beta\mu_I}{\alpha_d Z^+}\right)\right]. \tag{6}$$

Obviously, for all $\beta > 0$, we have

$$1 > r^+ > r^- > 0. \tag{7}$$

Because liquidity buyers assign equal probabilities to the two events $\{v = +\sigma\}$ and $\{v = -\sigma\}$, their dark pool crossing probability $(r^+ + r^-)/2$ is greater than informed traders' crossing probability r^-. In other words, correlated informed orders have a lower execution probability in the dark pool than relatively uncorrelated liquidity orders.

If the dark pool contains only liquidity orders (that is, $\beta = 0$), then any dark pool buy order has the execution probability

$$\bar{r} = \mathbb{E}\left[\min\left(1, \frac{Z^-}{Z^+}\right)\right]. \tag{8}$$

For our purposes, $\bar{r}$ measures the degree to which liquidity orders are balanced. Perfectly balanced liquidity orders correspond to $\bar{r} = 1$. For $\alpha_d = 0$, I define $r^+ = r^- = 0$.

The expected profits of an informed buyer on the exchange and in the dark pool are, respectively,

$$W_e = \sigma - S, \tag{9}$$

$$W_d = r^- \sigma. \tag{10}$$

I denote by c the delay cost of a generic liquidity buyer per unit of asset position. This buyer's per-unit net payoffs of deferring trade, trading on the exchange, and trading in the dark pool are, respectively,

$$X_0(c) = -c, \tag{11}$$

$$X_e = -S, \tag{12}$$

$$X_d(c) = -\frac{r^+ - r^-}{2}\sigma - c\left(1 - \frac{r^+ + r^-}{2}\right). \tag{13}$$

The terms on the right-hand side of (13) are the liquidity trader's adverse selection cost and delay cost in the dark pool, respectively. For $\beta > 0$, crossing in the dark pool implies a positive adverse selection cost because execution is more likely if a liquidity trader is on the side of the market opposite to that of informed traders. For $\beta = 0$, this adverse-selection cost is zero. For simplicity, in the remaining of the paper the net profits and delay costs of liquidity traders refer to profits and costs per unit of asset, unless otherwise specified. It is without loss of generality to focus on the venue decision for one unit of asset because, by risk neutrality, each trader's optimal venue choice is a corner solution with probability one.

From (9) and (12), $W_e - X_e = \sigma$. For all delay cost $c \leq \sigma$,

$$W_d - X_d(c) = \frac{r^+ + r^-}{2}\sigma + c\left(1 - \frac{r^+ + r^-}{2}\right) \leq \sigma = W_e - X_e. \tag{14}$$

That is, provided $c \leq \sigma$, the dark pool is more attractive to liquidity traders than to informed traders, relative to the exchange. In particular, (14) implies that a liquidity trader with a delay cost of σ (or a funding cost of $\gamma = 1$) behaves in the same way as an informed trader. In addition,

$$X_d(c) - X_0(c) = -\frac{r^+ - r^-}{2}\,\sigma + \frac{r^+ + r^-}{2}\,c. \tag{15}$$

So a liquidity trader with a funding cost of $\gamma = (r^+ - r^-)/(r^+ + r^-)$ is indifferent between deferring trade and trading in the dark pool.

Proposition 1. *There exists a unique threshold volatility $\bar{\sigma} > 0$ such that:*

1. *If $\sigma \le \bar{\sigma}$, then there exists an equilibrium $(\beta = 0, \alpha_d = \alpha_d^*, \alpha_e = 1 - \alpha_d^*)$, where $\alpha_d^* \in (0, G(1)]$ and μ_I^* solve*

$$G^{-1}(\alpha_d)(1-\bar{r}) = \frac{\mu_I}{\mu_I + (1-\alpha_d)\mu_z}, \tag{16}$$

$$\mu_I = \bar{\mu} F\left(\frac{(1-\alpha_d)\mu_z}{\mu_I + (1-\alpha_d)\mu_z}\,\sigma\right). \tag{17}$$

2. *If and only if $\sigma > \bar{\sigma}$, there exists an equilibrium $(\beta = \beta^*, \alpha_d = \alpha_d^*, \alpha_e = 1 - G(1))$, where β^*, $\alpha_d^* \in (0, G(1)]$, and μ_I^* solve*

$$r^- = 1 - \frac{(1-\beta)\mu_I}{(1-\beta)\mu_I + (1-G(1))\mu_z}, \tag{18}$$

$$\alpha_d = G(1) - G\left(\frac{r^+ - r^-}{r^+ + r^-}\right), \tag{19}$$

$$\mu_I = \bar{\mu} F\left(\frac{(1-G(1))\mu_z}{(1-\beta)\mu_I + (1-G(1))\mu_z}\,\sigma\right). \tag{20}$$

The proof of Proposition 1 is provided in Appendix C, but we outline its main intuition here. If the volatility σ is sufficiently low, the exchange spread is low; thus, the price-improvement benefit of the dark pool is lower than the cost of execution risk. In this case, informed traders avoid the dark pool (i.e. $\beta = 0$). The equilibrium is then determined by the marginal liquidity trader who is indifferent between trading on the exchange and trading in the dark pool, as well as by the marginal for-profit trader who is indifferent about whether to acquire the information.

If the volatility σ is sufficiently high, informed traders joint liquidity traders in the dark pool to avoid the higher exchange spread. Thus, $\beta \in (0, 1)$. In this case, the equilibrium is determined by three indifference conditions. First, informed traders must be indifferent between trading in either venue, as shown in (18). By (14), a liquidity trader with a delay cost of σ is also indifferent between the two venues. Thus, $\alpha_0 + \alpha_d = G(1)$ and $\alpha_e = 1 - G(1)$. The second indifference condition (19) then follows from (15). Here, the fraction α_0 of liquidity traders who delay trade must be strictly positive because informed traders introduce adverse selection into the dark pool. The third condition (20) says that the marginal for-profit trader is indifferent about whether to acquire the information.

Similarly, we can characterize an equilibrium for a market structure in which only the exchange is operating and the dark pool is absent. This exchange-only equilibrium, stated below, may also be interpreted as one in which a dark pool is open but no trader uses it.

Corollary 1. *With only an exchange and no dark pool, there exists an equilibrium in which $\beta^* = \alpha_d^* = 0$, and μ_I^* and $\alpha_e^* \in (1 - G(1), 1)$ solve*

$$\frac{\mu_I}{\mu_I + \alpha_e \mu_z} = G^{-1}(1 - \alpha_e) \tag{21}$$

$$\mu_I = \bar{\mu} F\left(\frac{\alpha_e \mu_z}{\mu_I + \alpha_e \mu_z}\,\sigma\right). \tag{22}$$

Equilibrium selection

The equilibria characterized in Proposition 1 need not be unique among all equilibria solving (16)-(17) and (18)-(20). For example, under the condition (63), both sides of (16) strictly increase in α_d. Similarly, both sides of (19) strictly increase in α_d, and both sides of (21) strictly decrease in α_e. Thus, given the absence of a single-crossing property, multiple equilibria may arise.[13]

I use stability as an equilibrium selection criterion, which allows me to compute the comparative statics of the selected equilibria. Among the equilibria characterized by Case 1 of Proposition 1, I select that with the smallest liquidity participation α_d^* in the dark pool among those with the property that, as α_d varies in the neighborhood of α_d^*, the left-hand side of (16) crosses the right-hand side from below.[14] Under the conditions of Proposition 1, this equilibrium exists and is robust to small perturbations.[15] Moreover, once α_d is determined in equilibrium, μ_I and β are uniquely determined, too, as shown in the proof of Proposition 1.

[13]One special condition that guarantees the uniqueness of the equilibrium in Case 1 of Proposition 1 is that the distribution function G of delay costs is linear. With a linear G, the condition (63) is also necessary for the existence of solutions to (16)-(17).

[14]Selecting the stable equilibrium corresponding to the smallest α_d^* is arbitrary but without loss of generality. As long as the selected equilibrium is stable, comparative statics calculated later follow through.

[15]If, for example, α_d^* is perturbed to $\alpha_d^* + \epsilon$ for sufficiently small $\epsilon > 0$, then the marginal liquidity trader has a higher cost in the dark pool than on the exchange, and therefore migrates out of the dark pool. Thus, α_d is "pushed back" to α_d^* and the equilibrium is restored. There is a symmetric argument for a small downward perturbation to $\alpha_d^* - \epsilon$. By contrast, if there is an equilibrium in which, as α_d varies, the left-hand side of (16) crosses the right-hand side from above, this equilibrium would not be stable to local perturbations.

Similarly, among equilibria characterized by Case 2 of Proposition 1, I select the one with the smallest liquidity participation α_d^* in the dark pool among those with the property that, as α_d varies in the neighborhood of α_d^*, the left-hand side of (19) crosses the right-hand side from below. In a market without a dark pool (Corollary 1), I select the equilibrium with the largest liquidity participation α_e^* on the exchange among those with the property that, as α_e varies in the neighborhood of α_e^*, the left-hand side of (21) crosses the right-hand side from below. By the argument given for Case 1 of Proposition 1, these selected equilibria exist and are stable.

3.3 Market characteristics and comparative statics

I now investigate properties of the equilibria characterized by Proposition 1. Proposition 2 and Proposition 3 below aim to answer two questions:

1. In a market with a dark pool and an exchange, how do market characteristics vary with the value σ of private information?
2. Given a fixed value σ of private information, how does adding a dark pool affect market behavior?

Proposition 2. *In the equilibrium of Proposition 1:*

1. *For $\sigma \leq \bar{\sigma}$, the dark pool participation rate α_d of liquidity traders, the total mass μ_I of informed traders, and the scaled exchange spread S/σ are strictly increasing in σ. The exchange participation rate $\alpha_e = 1 - \alpha_d$ of liquidity traders is strictly decreasing in σ. Moreover, α_d, μ_I, and S are continuous and differentiable in σ.*
2. *For $\sigma > \bar{\sigma}$, all of μ_I, $\beta\mu_I$, r^+, and S/σ are strictly increasing in σ, whereas α_d and r^- are strictly decreasing in σ. Moreover, β, α_d, μ_I, S, r^+, and r^- are continuous and differentiable in σ.*

In the equilibrium of Corollary 1, μ_I and S/σ are strictly increasing in σ, whereas α_e is strictly decreasing in σ. Moreover, α_e, μ_I, and S are continuous and differentiable in σ.

Proof. See Appendix C. □

Proposition 3. *In the equilibria of Proposition 1 and Corollary 1:*

1. *For $\sigma \leq \bar{\sigma}$, adding a dark pool strictly reduces the exchange participation rate α_e of liquidity traders and the total mass μ_I of informed traders. Adding a dark pool strictly increases the exchange spread S and the total participation rate $\alpha_e + \alpha_d$ of liquidity traders in either venue.*

2. *For $\sigma > \bar{\sigma}$, adding a dark pool strictly reduces α_e. Moreover, adding a dark pool strictly increases the exchange spread S if and only if, in the equilibrium of Proposition 1,*

$$r^- < 1 - \frac{\mu_I}{\mu_I + (1 - G(1 - r^-))\mu_z}. \tag{23}$$

It is sufficient (but not necessary) for (23) that

$$G''(\gamma) \leq 0 \text{ for all } 1 - \bar{r} \leq \gamma \leq 1 \text{ and } F(c) \to 1 \text{ for all } c > 0, \tag{24}$$

Proof. See Appendix C. □

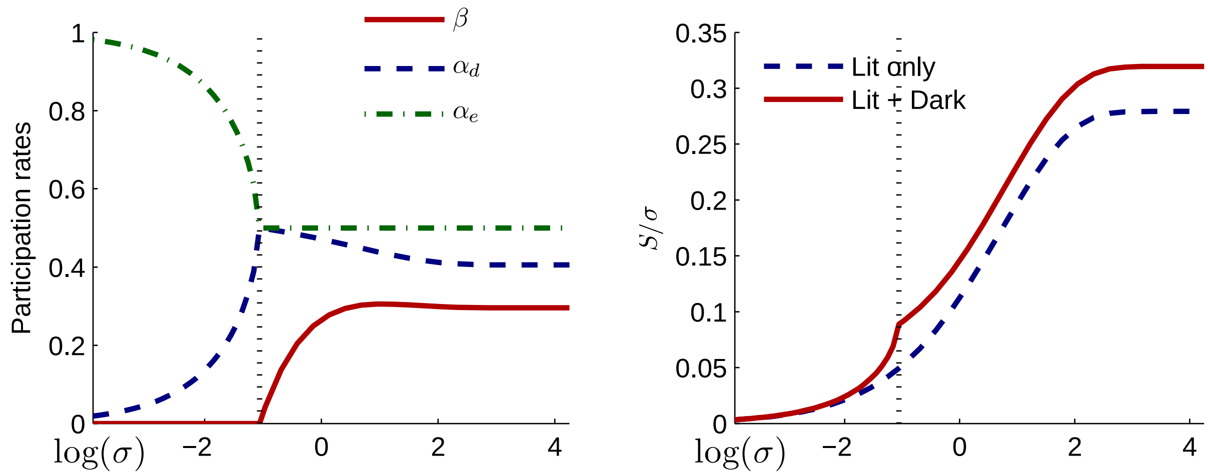
We now discuss the intuition and implications of Proposition 2 and Proposition 3 through numerical examples.

3.3.1 Participation rates and exchange spread

The left-hand side plot of Figure 3 shows the equilibrium participation rates in the exchange and the dark pool. For a small value of information, specifically if $\sigma \leq \bar{\sigma}$, informed traders trade exclusively on the exchange because the exchange spread is smaller than the cost of execution risk in the dark pool. An increase in σ widens the exchange spread, encouraging more liquidity traders to migrate into the dark pool. For $\sigma > \bar{\sigma}$, informed traders use both venues. We observe that informed dark pool participation rate β first increases in volatility σ and then decreases. The intuition for this non-monotonicity is as follows. Consider an increase in the value of information from σ to $\sigma + \epsilon$, for some $\epsilon > 0$. This higher value of information attracts additional informed traders. For a low β, the dark pool execution risk stays relatively low, and these additional informed traders prefer to trade in the dark pool, raising β. For sufficiently high β, however, informed orders cluster on one side of the dark pool and significantly reduce their execution probability. Thus, these additional informed traders send orders to the exchange, reducing β. Nonetheless, the total quantity $\beta\mu_I$ of informed traders in the dark pool is strictly increasing in σ. Finally, because informed participation in the dark pool introduces adverse selection, liquidity traders with low delay costs migrate out of the dark pool, leading to a decline in their dark pool participation rate α_d.

The right-hand side plot of Figure 3 shows the scaled exchange spread S/σ. Because a higher value σ of information encourages more for-profit traders to become informed, the scaled exchange spread S/σ increases in σ, whether a dark pool is present or not. For $\sigma \leq \bar{\sigma}$, adding a dark pool raises S/σ by diverting some liquidity traders, but none of the informed traders, off the exchange. For $\sigma > \bar{\sigma}$, adding a dark pool in

Figure 3: Participation rates and exchange spread. The left-hand side plot shows the equilibrium participation rates $(\beta, \alpha_d, \alpha_e)$ in a market with a dark pool. The right-hand side plot shows the scaled exchange spread S/σ. In both plots, the vertical dotted line indicates the threshold volatility $\bar{\sigma}$ at which the equilibrium of Proposition 1 changes from Case 1 to Case 2. Model parameters: $\mu_z = 60$, $\sigma_z = \sqrt{60}$, $\bar{\mu} = 20$, Z^+ and Z^- have $Gamma(30, 1)$ distributions, $G(s) = s/2$ for $s \in [0, 2]$, and $F(s) = 1 - e^{-s/2}$ for $s \in [0, \infty)$.



this example also increases the scaled spread S/σ because the dark pool diverts more liquidity traders than informed traders.

3.3.2 Price discovery

Now I turn to price discovery, by which I mean the extent to which the period-1 announcements (P_1, V_b, V_s) are informative of the fundamental asset value v. Since the market maker observes the volume (V_b, V_s), the closing price P_1 is

$$P_1 = \mathbb{E}[v \mid V_b, V_s]. \tag{25}$$

Because v is binomially distributed, its conditional distribution after period-1 trading is completely determined by its conditional expectation

$$\mathbb{E}[v \mid P_1, V_b, V_s] = \mathbb{E}[\mathbb{E}[v \mid V_b, V_s] \mid P_1] = P_1. \tag{26}$$

That is, all period-1 public information that is relevant for the asset value v is conveyed by the closing price P_1. As we will make precise shortly, the "closer" is P_1 to v, the better is price discovery.

Clearly, P_1 is uniquely determined by the log likelihood ratio

$$R_1 = \log \frac{\mathbb{P}(v = +\sigma \mid V_b, V_s)}{\mathbb{P}(v = -\sigma \mid V_b, V_s)} = \log \frac{\phi\left(Z^+ = \frac{1}{\alpha_e}[V_b - (1-\beta)\mu_I]\right) \cdot \phi\left(Z^- = \frac{1}{\alpha_e} V_s\right)}{\phi\left(Z^+ = \frac{1}{\alpha_e} V_b\right) \cdot \phi\left(Z^- = \frac{1}{\alpha_e}[V_s - (1-\beta)\mu_I]\right)}, \tag{27}$$

where ϕ is the probability density function of Z^+ and Z^-. We have also used the fact that the prior distribution $\mathbb{P}(v = +\sigma) = \mathbb{P}(v = -\sigma) = 0.5$.

Given R_1, the market maker sets the period-1 closing price

$$P_1 = \frac{e^{R_1} - 1}{e^{R_1} + 1}\,\sigma. \tag{28}$$

Conditional on P_1, a non-trader assigns the probability

$$Q_1 \equiv \mathbb{P}(v = +\sigma \mid V_b, V_s) = \frac{e^{R_1}}{e^{R_1} + 1} = \frac{1}{2}\left(\frac{P_1}{\sigma} + 1\right) \tag{29}$$

that the asset value is high.

Without loss of generality, I condition on $v = +\sigma$ and consider price discovery to be unambiguously "improved" if the probability distribution of R_1 is "increased," in the sense of first-order stochastic dominance. Complete revelation of $v = +\sigma$ corresponds to $R_1 = \infty$ almost surely.

In general, we need to know the functional form of the density $\phi(\cdot)$ in order to explicitly calculate R_1, P_1, and Q_1. However, since the distribution of Z^+ and Z^- is infinitely divisible, Z^+ and Z^- can be expressed as the sums of i.i.d. random variables. Further, we can always take an example in which, by the central limit theorem, the density $\phi(\cdot)$ is approximated by $Normal(0.5\mu_z, 0.5\sigma_z^2)$ when μ_z and σ_z^2 are sufficiently large.[16] Substituting into (27) the normal density function, we can approximate R_1 by

$$R_1^{normal} = \frac{2(1-\beta)\mu_I}{\alpha_e^2 \sigma_z^2}(V_b - V_s), \tag{30}$$

which is the counterpart of R_1 under the normal distribution.[17] Given $v = +\sigma$, $V_b - V_s$ has a distribution close to that of $Normal((1-\beta)\mu_I, \alpha_e^2\sigma_z^2)$, so R_1 is has a distribution

[16] We can show this approximation as follows. Fix a small $\delta > 0$ such that $m = \mu_z/\delta$ is an integer. By infinite divisibility, Z^+ can be represented as the sum $\sum_{i=1}^m Z_{im}^+$, where $\{Z_{im}^+\}$ are i.i.d. random variables with mean δ and variance $\delta\sigma_z^2/\mu_z$. Fixing δ, the central limit theorem implies that the distribution of Z^+ is approximately normal when m is large, that is, when μ_z and σ_z^2 are large.

[17] In the calculation of (30), I have used the central limit theorem and the fact that $\phi(\cdot)$ and the normal density are positive and continuous in $[0, \infty)$.

close to that of

$$Normal\left(2\left(\frac{(1-\beta)\mu_I}{\alpha_e\sigma_z}\right)^2, 4\left(\frac{(1-\beta)\mu_I}{\alpha_e\sigma_z}\right)^2\right) \sim Normal\left(2I(\beta,\alpha_e)^2, 4I(\beta,\alpha_e)^2\right), \tag{31}$$

where

$$I(\beta,\alpha_e) \equiv \frac{(1-\beta)\mu_I}{\alpha_e\sigma_z} \tag{32}$$

is the "signal-to-noise" ratio, which is the mass of informed orders on the exchange ("signal") divided by the standard deviation of the imbalance of liquidity orders on the exchange ("noise"). Naturally, $I(\beta,\alpha_e)$ is increasing in the scaled exchange spread S/σ.

Figure 4 plots the distribution function of R_1, under normal approximation, with and without a dark pool. The value σ of information is set to be the threshold value $\bar{\sigma}$, so that $\beta = 0$ in the equilibria with a dark pool as well as the equilibria without a dark pool. By Proposition 3, adding a dark pool strictly increases the scaled spread S/σ and hence the signal-to-noise ratio $I(\beta,\alpha_e)$. With a dark pool, the conditional distribution of R_1 has a higher mean, but also a higher variance. For most realizations of R_1, and on average, adding a dark pool decreases the cumulative distribution of R_1 and leads to a more precise inference of v. Nonetheless, adding a dark pool may increase the cumulative distribution of R_1, thus harming price discovery, when the realization of R_1 is sufficiently low.

The price-discovery effect of the dark pool is further illustrated in Figure 5. The left-hand plot of Figure 5 shows the probability density function of Q_1, under normal approximation, with and without a dark pool. As in Figure 4, adding a dark pool shifts the probability density function of Q_1 to the right, improving price discovery on average.[18] Nonetheless, the dark pool increases the probability of extremely low realizations of Q_1, harming price discovery in these unlikely events. The right-hand plot of Figure 5 shows how Q_1 depends on the imbalance $Z = Z^+ - Z^-$ of liquidity order flow. Again, for most realizations of Z, adding the dark pool increases Q_1, improving price discovery. For unlikely low realizations of Z, adding the dark pool reduces Q_1, thus harming price discovery. That is, when the trading interests of liquidity traders are sufficiently large and opposite in direction to the informed, adding the dark pool can exacerbate the "misleading" inference regarding the asset value. Because liquidity trading interests are balanced in expectation, such misleading events are rare, and the

[18] We can analytically show that the expectation $\mathbb{E}[Q_1]$ under normal approximation is increasing in the signal-to-noise ratio $I(\beta,\alpha_e)$.

Figure 4: Distribution functions of R_1, under normal approximation, with and without a dark pool. The true dividend is the threshold value $+\bar{\sigma}$ and other parameters are those of Figure 3.

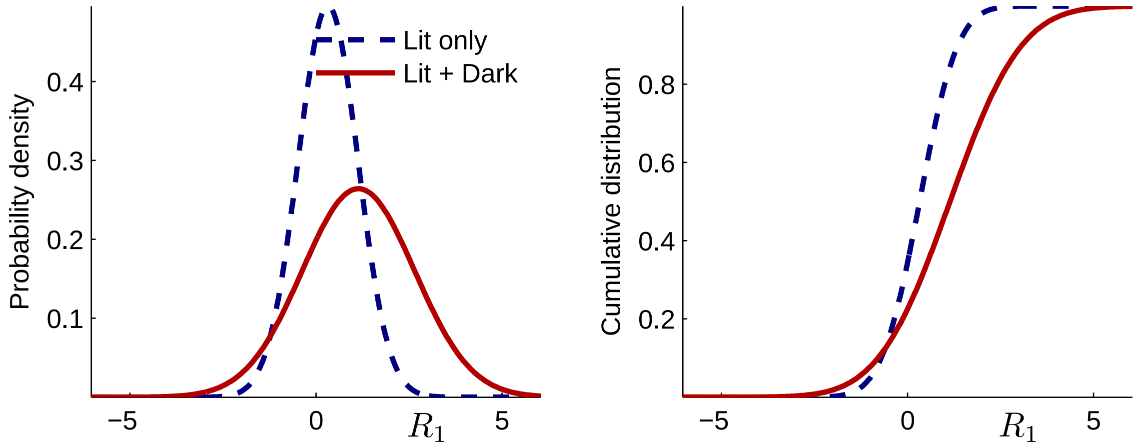
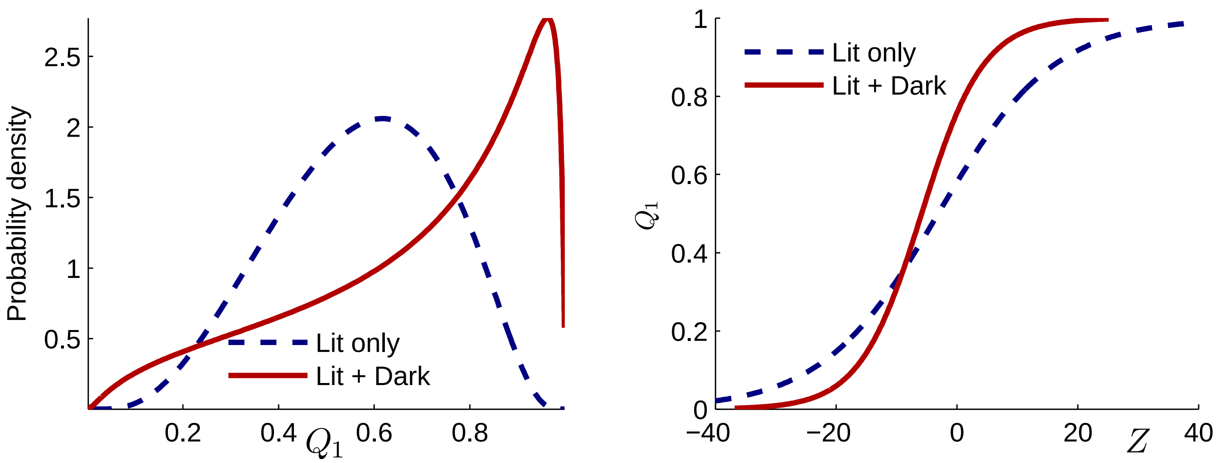


Figure 5: The left-hand plot shows the probability density function of Q_1, under normal approximation, with and without a dark pool. The right-hand plot shows how Q_1 depends on the order imbalance $Z = Z^+ - Z^-$ of liquidity order flow. Model parameters are those of Figure 4.



dark pool is normally beneficial for price discovery.

We observe that the practical interpretation of price discovery depends largely on the horizon of information. Because trading is frequent and fast (with the exception that large orders can take days to fill), dark pools are most likely to concentrate short-term information, rather than long-term information, onto the exchange. Short-term information can be fundamental (such as merger announcements, earnings reports, or macroeconomic news) or technical (such as the order flows of large institutions). Moreover, when both short-term investors and long-term investors are present, it is natural to interpret the former as informed and the latter as uninformed. Under this interpretation, the results of this paper suggest that dark pools are more attractive to long-term investors than to short-term investors, relative to the exchange.

3.3.3 Dark pool market share

I now calculate the dark pool market share, i.e. the proportion of trading volume handled by the dark pool. The market share of the dark pool is a direct empirical measure of dark pool activity. I assume that once the dividend v is announced in period 2, informed traders who have not yet traded leave the market, because they will not be able to trade profitably. When calculating the exchange volume, I also include the transactions of liquidity traders in period 2. Thus, the expected trading volumes in the dark pool, on the exchange, and in both venues are, respectively,

$$V_d = \beta\mu_I r^- + \alpha_d\mu_z \frac{r^+ + r^-}{2}, \tag{33}$$

$$V_e = (1-\beta)\mu_I + \alpha_e\mu_z + \alpha_d\mu_z\left(1 - \frac{r^+ + r^-}{2}\right) + \alpha_0\mu_z, \tag{34}$$

$$V = V_e + V_d = \mu_z + \mu_I(1 - \beta + \beta r^-). \tag{35}$$

By Proposition 2, these volumes are differentiable in the volatility σ in each of the two intervals $[0, \bar{\sigma}]$ and $(\bar{\sigma}, \infty)$.

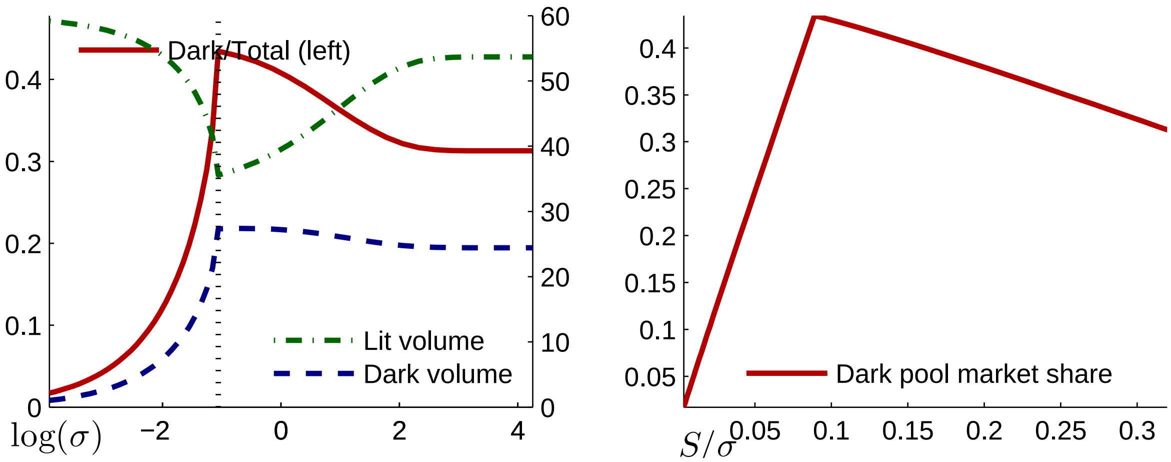
For $\sigma \leq \bar{\sigma}$, the dark pool volume, $V_d = \alpha_d\mu_z\bar{r}$, is increasing in the volatility σ, by Proposition 2. In particular, as $\sigma \to 0$, the dark pool participation rate α_d of liquidity traders and the dark pool market share V_d/V converge to zero. For a sufficiently small $\sigma < \bar{\sigma}$, therefore,

$$\frac{d(V_d/V)}{d\sigma} = \frac{d}{d\sigma}\left(\frac{\alpha_d\mu_z\bar{r}}{\mu_z + \mu_I}\right) = \frac{\mu_z\bar{r}}{\mu_z + \mu_I}\cdot\frac{d\alpha_d}{d\sigma} - \frac{\alpha_d\mu_z\bar{r}}{(\mu_z + \mu_I)^2}\cdot\frac{d\mu_I}{d\sigma} > 0,$$

where the inequality follows from the fact that $\lim_{\sigma\to 0} d\alpha_d/d\sigma > 0$ (shown in the proof of Proposition 2). That is, if the volatility σ is sufficiently low, then the dark pool market share V_d/V is increasing in σ. For $\sigma \leq \bar{\sigma}$, because the total volume $V = \mu_z + \mu_I$ is increasing in σ, the dark pool market share is increasing in the total volume, as illustrated in the left-hand plot of Figure 6.

Figure 6 further suggests that, as the volatility σ increases beyond $\bar{\sigma}$, the exchange volume V_e can increase substantially, but the dark pool volume V_d may only increase mildly or even decline. Thus, the dark pool market share can decrease in volatility σ for sufficiently large σ, creating a hump-shaped relation between volatility and the dark pool market share. The model also generates a similar relation between the scaled

Figure 6: Expected trading volume on the exchange and in the dark pool. The left-hand plot shows the volume in the two venues and the market share of the dark pool. The right-hand plot shows the dark pool market share as a function of the scaled spread S/σ. The vertical dotted line corresponds to the threshold volatility $\bar{\sigma}$. Parameters are those of Figure 3.



spread S/σ and the dark pool market share V_d/V, as shown on the right-hand plot of Figure 6.

4 Dark Pools as Nondisplayed Limit Order Books

So far we have studied a dark pool that crosses orders at the midpoint of the exchange bid and ask. In this section, I model a dark pool that operates as a nondisplayed limit order book, where execution prices depend on submitted limit orders, as described in Section 2. Aside from confirming the basic intuition of Section 3, this section offers additional insights regarding the impact of dark pool mechanisms on the participation incentives of informed traders.

Although limit-order dark pools may execute orders at prices other than the midpoint, such price discretion is often limited by "best-execution" regulations. In the United States, the Order Protection Rule, also known as the "trade through" rule, stipulates that transaction prices in any market center—including dark pools, ECN, and broker-dealer internalization—cannot be strictly worse than the prevailing national best bid and offer (NBBO).[19] For example, if the current best bid is \$10 and the best ask is \$10.50, then the transaction price in any market center must be in the interval $[\$10, \$10.50]$. More recently, regulators have also proposed a stricter "trade-at" rule.

[19] In Europe, MiFID uses a decentralized best-execution rule, by which investment firms decide whether an execution works for the best interest of investors.

Under a trade-at rule, execution prices in dark pools must be strictly better than the best bid or offer on all displayed venues, including exchanges. For example, the Joint CFTC-SEC Advisory Committee (2011) recommends that the SEC consider "its rule proposal requiring that internalized or preferenced orders only be executed at a price materially superior (e.g. 50 mils [0.5 cent] for most securities) to the quoted best bid or offer."

I now describe and solve a simple model of a limit-order dark pool that operates under a trade-at rule. The dark pool executes orders by price priority, and I model its trading mechanism as a double auction. The dark-pool execution price, p^*, is determined such that the aggregate limit buy orders (i.e. demand) at p^* is equal to the aggregate sell limit orders (i.e. supply) at p^*. Moreover, I model the effect of a trade-at rule by assuming that transaction prices in the dark pool must be within the interval $[-xS, xS]$, where $S > 0$ is the exchange spread and $x \in [0, 1]$ captures the strictness of the trade-at rule. The trade-through rule currently applied in the United States corresponds to $x = 1$, indicating a mandatory price improvement of zero. A midpoint-matching mechanism corresponds to $x = 0$, indicating a price improvement of the entire effective spread S. With the exception of this trade-at rule, the model of this section is identical to that of Section 3. Proposition 4 below characterizes an equilibrium that is analogous to Case 1 of Proposition 1.[20] This result sheds light on how the trade-at rule affects the dark pool participation of informed traders.

Proposition 4. *In a market with an exchange and a dark pool that implements a double auction, there exists a unique threshold volatility $\bar{\sigma}(x) > 0$ with the property that, for any $\sigma \leq \bar{\sigma}(x)$, there exists an equilibrium ($\beta = 0, \alpha_d = \alpha_d^*, \alpha_e = 1 - \alpha_d^*$), where $\alpha_d^* \in (0, G(1)]$ and μ_I^* solve*

$$\left[G^{-1}(\alpha_d) - \frac{xS}{\sigma}\right] \cdot (1 - \bar{r}_x) = \frac{\mu_I}{\mu_I + (1 - \alpha_d)\mu_z}, \tag{36}$$

$$\mu_I = \bar{\mu} F\left(\frac{(1 - \alpha_d)\mu_z}{\mu_I + (1 - \alpha_d)\mu_z}\,\sigma\right). \tag{37}$$

In this equilibrium with a fixed x:

1. *If $c \in [0, xS)$, a liquidity buyer (resp. seller) with a delay cost of c quotes a limit*

[20] For tractability reasons, I have not characterized an equilibrium in which some informed traders send orders to the limit-order dark pool. A modeling challenge with informed participation in the limit-order dark pool is to calculate the expected loss of liquidity traders, conditional on order execution at each possible price in the interval $[-xS, xS]$, not only the midpoint. Boulatov and George (2010) model a nondisplayed market in which informed traders submit demand schedules (i.e. limit orders). Their model is tractable partly because their uninformed traders are noise traders and hence do not internalize the costs of trading against informed traders. By contrast, endogenous venue selection of liquidity traders is a key modeling objective of this paper.

price of c (resp. $-c$) in the dark pool. If $c \in [xS, G^{-1}(\alpha_d^*)\sigma]$, then a liquidity buyer (resp. seller) with a delay cost of c quotes a limit price of xS (resp. $-xS$) in the dark pool. Liquidity traders with delay costs higher than $G^{-1}(\alpha_d^*)\sigma$ trade on the exchange.

2. *The dark pool execution price is given by (79) in the appendix.*

3. *The dark pool participation rate α_d of liquidity traders, the mass μ_I of informed traders, and the scaled exchange spread S/σ are all strictly increasing in the value σ of information.*

Moreover, for $x \in (0, 1)$, the volatility threshold $\bar{\sigma}(x)$ is strictly decreasing in x.

Proof. See Appendix C. □

The equilibrium of Proposition 4 with a limit-order dark pool is qualitatively similar to the equilibrium characterized in Case 1 of Proposition 1. The equilibrium is determined by the marginal liquidity trader who is indifferent between the two venues, shown in (36), and the marginal for-profit trader who is indifferent about whether to acquire the information, shown in (37). If multiple equilibria exist, I select the equilibrium with the lowest α_d^* among those with the property that, as α_d varies in a neighborhood of α_d^*, the left-hand side of (36) crosses the right-hand side from below. The expressions of $\bar{\sigma}(x)$ and p^* in equilibrium are provided in Appendix C.

Naturally, in equilibrium liquidity traders who have higher delay costs submit more aggressive orders (i.e. buy orders with higher limit prices and sell orders with lower limit prices). Moreover, because a liquidity buyer's order is infinitesimal and has zero impact on the execution price p^*, she wishes to use a "truth-telling" strategy, that is, to submit a buy order whose limit price is equal to her delay cost.[21] If her decay cost $c < xS$, the trade-at rule is not binding, so she submit a dark pool buy order with the limit price c. If $c \geq xS$, the trade-at rule becomes binding at the price xS, so the liquidity buyer selects the highest limit price allowed, xS. In equilibrium, a strictly positive mass of liquidity buyers set the limit price xS and are rationed with a positive probability. When the delay cost c is sufficiently high, the liquidity buyer trades on the exchange in order to avoid the risk of being rationed at the price xS. The intuition for a liquidity seller is symmetric.

Proposition 4 further reveals that the trade-at rule has a material effect for the participation of informed traders in the dark pool. Because $\bar{\sigma}(x)$ is decreasing in x,

[21]This strategy is reminiscent of the truth-telling strategy of MacAfee (1992), who considers a double auction with finitely many buyers and sellers. The double auction here has the institutional restriction that transaction prices are bounded by the trade-at rule.

the stricter is the trade-at rule, the less attractive is the dark pool to informed traders. The intuition is as follows. If an informed buyer were to deviate to the dark pool, she would select the most aggressive permissible limit price, xS, in order to maximize her execution probability. Although she would be rationed at the price xS, she would only compete with those liquidity traders who have a delay cost of xS or higher. The lower is x, the less scope there is for the informed trader to "step ahead of the queue" and gain execution priority. In particular, a midpoint dark pool with $x = 0$ has the greatest effectiveness in discouraging informed traders to participate.

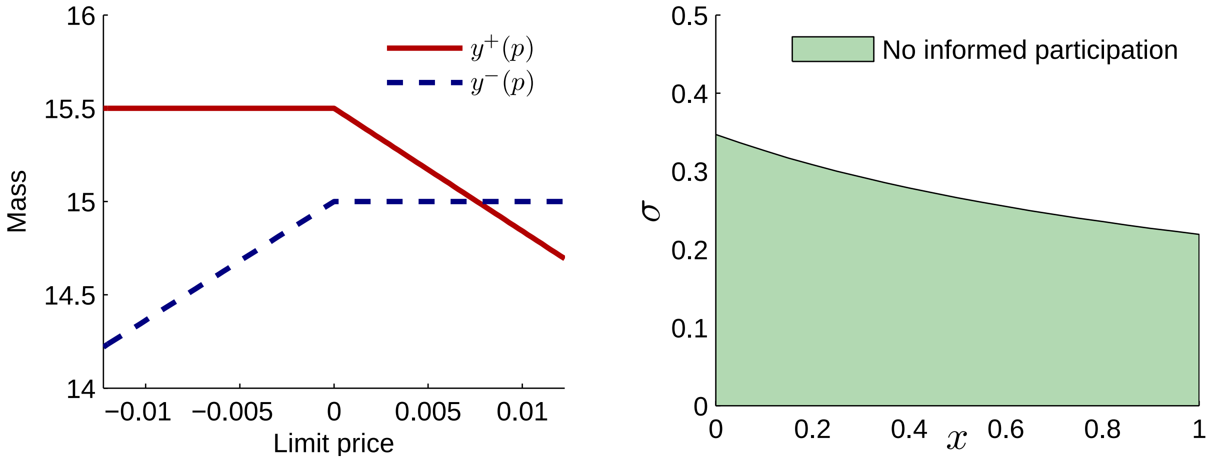
The left-hand plot of Figure 7 shows the dark pool orders in the equilibrium of Proposition 4. In this example, $x = 0.8$, so the dark pool provides a price improvement equal to 20% of the exchange spread S. In this example, about 95% of liquidity traders in the dark pool set the most aggressive limit price, $\pm xS$. The dark pool transaction price in this case is about 0.007. The right-hand plot of Figure 7 shows that the volatility threshold $\bar{\sigma}(x)$ is strictly decreasing in x. With midpoint crossing ($x = 0$), informed traders avoid the dark pool if the value σ of information is lower than about 0.35. Under the current trade-through rule ($x = 1$), this volatility threshold is reduced to about 0.22.

The effect of the trade-at rule on informed participation in dark pools complements prior fairness-motivated arguments, which suggest that displayed orders should have strictly higher priority than nondisplayed orders at the same price (Joint CFTC-SEC Advisory Committee, 2011). Proposition 4 predicts that implementing a trade-at rule is likely to reduce informed participation in dark pools. It also predicts that dark pools operating as limit order books are more likely to attract informed traders and impatient liquidity traders than dark pools crossing at the midpoint.

My model of a limit-order dark pool is related to and complements that of Buti, Rindi, Wen, and Werner (2011), who study the effect of tick size for market quality. In their model, a limit order book with a subpenny tick size is similar to a dark pool studied in this section. Since their model allows traders to post limit orders on the displayed market, it generates predictions on quote depths, which are not offered in my model. On the other hand, my model focuses on asymmetric information and price discovery, which are absent in their model. A desirable (and nontrivial) extension of my price-discovery model is to fully allow limit orders in both the exchange and the dark pool, and this extension is left for future research. The extensive literature on *displayed* limit order books is surveyed by Parlour and Seppi (2008).

The model of this section also differs from existing studies of nondisplayed markets that operate alone. For example, Boulatov and George (2010) model how informed

Figure 7: A dark pool as a nondisplayed limit order book. The left-hand plot shows the aggregate limit orders in the dark pool, where $y^+(p)$ and $y^-(p)$ denotes the demand schedule and supply schedule, respectively. The right-hand plot shows the range of σ for which the equilibrium of Proposition 4 exists, that is, informed traders avoid the dark pool. Model parameters are those of Figure 3. The left-hand plot also uses $x = 0.8$, $\sigma = \bar{\sigma}(0.8) = 0.236$, and realizations $Z^+ = 31$ and $Z^- = 30$.



traders provide liquidity through limit orders when their demand schedules (i.e. limit orders) are hidden. This nondisplayed market, they conclude, encourages informed traders to trade more aggressively on their information, hence improving price discovery, relative to a displayed market. Hendershott and Jones (2005) empirically study price discovery for exchange-traded funds (ETFs) when Island ECN stopped displaying its limit orders. Since Island ECN was the *dominant* market for affected ETFs, it differed from today's equity dark pools, which operate alongside exchanges.

5 Dynamic Trading

This short section generalizes the basic intuition of Section 3 to a dynamic market. Under natural conditions, all equilibria have the property that, after controlling for delay costs, an informed trader prefers the exchange to the dark pool, relative to a liquidity trader.

Time is discrete, $t \in \{1, 2, 3, \ldots\}$. As before, an asset pays an uncertain dividend v that is $+\sigma$ or $-\sigma$ with equal probabilities. The dividend is announced at the beginning of period $T \geq 2$, where T is deterministic, and paid at the end of period T. The trading game ends immediately after the dividend payment.

In each period before the dividend payment, a new set of informed traders and liquidity traders arrive. To simplify the analysis, I drop endogenous information acquisition in this section. (Equivalently, it costs zero to acquire information.) The mass of informed traders arriving in period t, $\mu_I(t) > 0$, is deterministic. Informed traders observe the dividend v and trade in the corresponding direction. The mass of liquidity buy orders and the mass of liquidity sell orders arriving in period t are $Z^+(t) > 0$ and $Z^-(t) > 0$, respectively, with commonly known probability distributions. The public does not observe v or the realizations of $Z^+(t)$ or $Z^-(t)$.

As before, a lit exchange and a dark pool operate in parallel. Both venues are open in all periods. At the beginning of period t, the exchange market maker posts a bid price B_t and an ask price A_t. Any order sent to the exchange is immediately executed at the bid or the ask. After execution of exchange orders in each period, the market maker announces the exchange buy volume and the exchange sell volume. The public information $\mathcal{F}_t$ at the beginning of period t consists of all exchange announcements prior to, but not including, period t. Thus, the conditional distribution of asset value v at the beginning of period t is represented by the likelihood ratio

$$R_t = \frac{\mathbb{P}_t(v = +\sigma)}{\mathbb{P}_t(v = -\sigma)}, \tag{38}$$

where $\mathbb{P}_t$ denotes the conditional probability based on $\mathcal{F}_t$. By construction, $R_0 = 1$. The public's conditional expectation of the asset value at the beginning of period t is therefore

$$V(R_t) = \sigma(\mathbb{P}_t(v = +\sigma) - \mathbb{P}_t(v = -\sigma)) = \frac{R_t - 1}{R_t + 1}\,\sigma. \tag{39}$$

The dark pool executes orders in each period, simultaneously with the execution of exchange orders. The dark pool implements a double auction with a trade-at rule, as in Section 4. Midpoint crossing, which offers a price improvement of the exchange spread, is a special case of this double auction. Liquidity traders differ from each other in their delay costs, as in Section 3. If a liquidity trader of cost type γ does not trade in period t, then she incurs a delay cost of $c(\gamma; R_t)$ in period t, where $c(\gamma; R_t)$ is strictly increasing in γ for all R_t. A trader only incurs delay costs after she arrives.

To control for traders' characteristics other than information, I make the additional assumption that informed traders also incur positive delay costs, before they execute their orders. A type-γ informed trader incurs the delay cost $c(\gamma; R_t)$ in period t if she fails to execute her order in that period. In practice, this cost may come from the opportunity cost of capital. Thus, a type-γ informed buyer (resp. seller) and a type-γ liquidity buyer (resp. seller) differ only in their information about v.

I now fix a cost type $\gamma \geq 0$ and compare the venue choice of a type-γ informed buyer with that of a type-γ liquidity buyer. For any (R_t, t), I let

$$W_e(R_t, t) = \sigma - A_t \tag{40}$$

$$X_e(R_t, t) = V(R_t) - A_t \tag{41}$$

be the payoffs of a type-γ informed buyer and a type-γ liquidity buyer, respectively, for trading immediately on the exchange. These payoffs do not depend on the cost type γ because exchange execution incurs no delays. I let $W_d(R_t, t; \gamma)$ and $X_d(R_t, t; \gamma)$ be the corresponding continuation values of entering an order in the dark pool. Finally, I let $W(R_t, t; \gamma)$ and $X(R_t, t; \gamma)$ be the continuation values of the informed buyer and liquidity buyer, respectively, at the beginning of period t, before they make trading decisions. For $t = T$, $W(R_T, T; \gamma) = X(R_T, T; \gamma) = 0$. For $t < T$, the Bellman Principal implies that

$$W(R_t, t; \gamma) = \max\left[W_e(R_t, t), W_d(R_t, t), \mathbb{E}_t(W(R_{t+1}, t+1; \gamma))\right], \tag{42}$$

$$X(R_t, t; \gamma) = \max\left[X_e(R_t, t), X_d(R_t, t), \mathbb{E}_t(X(R_{t+1}, t+1; \gamma))\right], \tag{43}$$

where the three terms in the $\max(\,\cdot\,)$ operator represents a trader's three choices: sending her order to the exchange, sending her order to the dark pool, and delaying trade.

The following proposition characterizes equilibrium conditions under which, controlling for delay costs, the liquidity-versus-informed payoff difference $X_d(R_t, t; \gamma) - W_d(R_t, t; \gamma)$ in the dark pool is at least as high as the corresponding payoff difference $X_e(R_t, t) - W_e(R_t, t)$ on the exchange. It is in this "difference-in-difference" sense that the dark pool is more attractive to liquidity traders, and that the exchange is more attractive to informed traders.

Proposition 5. *In any equilibrium, if $W_d(R_t, t; \gamma) \geq \mathbb{E}_t[W(R_{t+1}, t+1; \gamma)]$, then*

$$X_d(R_t, t; \gamma) - W_d(R_t, t; \gamma) \geq X_e(R_t, t) - W_e(R_t, t). \tag{44}$$

Proof. See Appendix C. □

Proposition 5 reveals that all equilibria must satisfy the restriction (44), provided that a informed buyer weakly prefers using the dark pool to delaying trade. The intuition for this result is as follows. Because the exchange guarantees to execute all buy orders at the same price A_t, the exchange payoff difference, $X_e(R_t, t) - W_e(R_t, t)$, reflects only the value of private information. The dark pool payoff difference $X_d(R_t, t; \gamma) - W_d(R_t, t; \gamma)$, by contrast, reflects both the value of information and the execution risk.

Compared with a liquidity buyer, an informed buyer in the dark pool is less likely to fill her order and, conditional on a trade, more likely to pay a higher price. This execution risk is costly for the informed buyer in equilibrium as long as she prefers dark pool trading to delaying, as captured by $W_d(R_t, t; \gamma) \geq \mathbb{E}_t[W_d(R_{t+1}, t+1; \gamma)]$. I isolate this dark pool execution risk from the value of information by taking the "difference-in-difference" of payoffs in (44).

Appendix B explicitly solves a dynamic equilibrium in a setting where traders arrive in Poisson times.

6 Implications and Discussions

This section discusses some implications of my results, both in light of recent empirical evidence and in relation to the current policy debate on the impacts of dark pools on price discovery and liquidity. The discussion follows two organizing questions. First, what are the relations between dark pool market share and observable market characteristics? Second, what are the impacts of dark pool trading on price discovery and liquidity? For each question, I discuss empirical implications of the model and put them in the context of related empirical evidence.

6.1 Determinants of dark pool market share

Prediction 1. *All else equal, dark pool market share is lower if the execution probability of dark pool orders is lower.*

Prediction 2. *All else equal, if the level of adverse selection (or volatility) is low, then dark pool market share is increasing in adverse selection (or volatility). If the level of adverse selection (or volatility) is high, then dark pool market share can be decreasing in adverse selection (or volatility).*

Prediction 3. *All else equal, informed participation in dark pools is higher if volatility is higher. Informed participation is higher in dark pools that allow more discretion in execution prices, compared with dark pools that execute orders at the exchange midpoint.*

Prediction 4. *All else equal, dark pool market share is lower for trading strategies relying on shorter-term information. The use of dark pools is also lower for trading strategies that trade multiple stocks simultaneously, compared with strategies that trade individual stocks one at a time.*

Prediction 1 follows from the results of Section 3 and Section 4. A lower execution probability, captured by a lower $\bar{r}$ or $\bar{r}_x$, discourages both types of traders from participating in the dark pool.[22] Using daily data collected by SIFMA from eleven anonymous dark pools in 2009, Buti, Rindi, and Werner (2011a) find that dark pool market share is negatively related to the order imbalance as a percentage of total volume and to the absolute depth imbalance on lit venues. Prediction 1 is consistent with their findings.[23]

Prediction 2 suggests that dark pool market share can be increasing or decreasing in the level of adverse selection (or volatility), depending on whether σ, the value of private information, is below or above the threshold $\bar{\sigma}$. Using transaction data in two block-crossing dark pools (Liquidnet and Posit), Ready (2012) finds that institutions are less likely to route orders to dark pools when the level of adverse selection is higher. In different samples, Buti, Rindi, and Werner (2011a) and Ye (2010) find that dark pool market shares are lower when volatilities and spreads are higher. To the extent that at least some informed traders participate in dark pools in practice,[24] and that volatilities and spreads are positively correlated with adverse selection, these findings are broadly consistent with (the latter half of) Prediction 2.

Prediction 3 suggests that dark pool orders are more informative on average when information asymmetry is severe. This prediction is consistent with recent evidence documented by Nimalendran and Ray (2012) in an anonymous dark pool. They infer the trading direction of each dark pool transaction by comparing the execution price with the prevailing market midpoint. A trading strategy that follows the directions of dark pool orders is profitable when spreads are wide but not profitable when spreads are narrow. To the extent that exchange spreads are proxy measures for adverse selection, Prediction 3 is consistent with their results. Prediction 3 also suggests that orders in limit-order dark pools are more informative than those in midpoint dark pools. To my knowledge, the latter half of Prediction 3 is not yet tested in the data.

Prediction 4 provides strategy-level implications on dark pool activity. Strategies relying on shorter-term information have higher execution risks in dark pools because relevant information can become stale sooner. Related to this prediction, Ready (2012) finds that the usage of block-crossing dark pools is lower for institutions with higher turnover, which is consistent with the notion that short-term strategies are best implemented in venues that guarantee execution. Because dark pools cannot guarantee the

[22] This relation can be analytically proved for $\sigma < \bar{\sigma}$ in Proposition 1 and for $\sigma < \bar{\sigma}(x)$ in Proposition 4.

[23] Related, Ye (2010) constructs a proxy for execution probability in eight dark pools from their SEC Rule 605 reports, and studies the relationship between non-execution probability and market characteristics (e.g. price impacts and effective spreads). He does not examine how non-execution probability relates to the market share of dark pools. For more details of Rule 605 of Reg NMS, see http://www.finra.org/Industry/Regulation/Guidance/SECRule605/.

[24] Recall from the model of Section 3 that if $\sigma > \bar{\sigma}$, then some informed traders participate in the dark pool.

simultaneous execution of trades in multiple stocks, we also expect dark pools to be less attractive for strategies tracking stock indices or "arbitraging" perceived mispricing among similar securities. For these strategies, partial execution in dark pools can be particularly costly.

6.2 Effects of dark pools on price discovery and liquidity

Prediction 5. *All else equal, a higher dark pool market share is associated with higher order informativeness, wider spreads, and higher price impacts of trades on the exchange.*

Prediction 6. *All else equal, a higher dark pool market share increases the correlation of volumes across different stocks in lit exchanges. This cross-stock volume correlation is lower in dark pools than in lit exchanges.*

Prediction 7. *All else equal, dark pool execution implies a positive adverse-selection cost, in that shares bought in dark pools tend to have low short-term returns and that shares sold in dark pools tend to have high short-term returns. This cost, however, is lower than the exchange spread at the time of execution.*

Proposition 3 provides sufficient conditions under which Prediction 5 holds. Prediction 5 is consistent with empirical evidence from Degryse, de Jong, and van Kervel (2011), Nimalendran and Ray (2012), Jiang, McInish, and Upson (2011), and Weaver (2011), but not Buti, Rindi, and Werner (2011a) or O'Hara and Ye (2011). In Dutch equity markets, Degryse, de Jong, and van Kervel (2011) find that higher market shares of dark trading—including dark pools and over-the-counter markets—are associated with higher price impacts, higher quoted spreads, higher realized spreads, and smaller depths on lit markets. Similarly, in U.S. equity markets, Jiang, McInish, and Upson (2011) find that off-exchange (dark) order flows are less informative than exchange (lit) order flows, after adjusting for trading volumes in dark and lit markets. Their results also indicate that exchange order flows become more informative as off-exchange order flows increase. Weaver (2011) finds that higher levels of off-exchange trading in the U.S. are associated with wider spreads, higher price impacts, and higher volatilities. Using transaction data in an anonymous dark pool, Nimalendran and Ray (2012) document that following dark-pool transactions, bid-ask spreads tend to widen and price impacts tend to increase, especially if the relative bid-ask spreads are already high. By contrast, Buti, Rindi, and Werner (2011a) find that higher dark pool trading activity tends to be associated with lower spreads and lower return volatilities, which suggest a better market quality. O'Hara and Ye (2011) also conclude that higher fragmentation of trading is associated with faster execution, lower transaction costs, and more efficient prices. Given the wide variety of data samples used in these studies and the difficulty in completely correcting for endogeneity, we should interpret these conflicting results with caution.

Prediction 6 can be viewed as the mirror image of Prediction 4. Since dark pools are less attractive to strategies that execute multiple stocks simultaneously, those strategies should have higher concentration in lit venues than in dark pools. Consequently, the volume correlation across stocks should be higher in lit venues than in dark pools. To my knowledge, this prediction is not yet tested in the data.

Finally, Prediction 7 on the adverse selection in dark pools is consistent with Sofianos and Xiang (2011), who find that dark pools that have higher execution probabilities also have more severe adverse selection (that is, more "toxic"). Næs and Odegaard (2006) provide anecdotal evidence that filled orders in a dark pool are subject to short-term losses. Mittal (2008) and Saraiya and Mittal (2009) emphasize that short-term adverse selection in dark pools can reduce execution quality of institutional investors. Conrad, Johnson, and Wahal (2003), Brandes and Domowitz (2010), and Domowitz, Finkelshteyn, and Yegerman (2009) examine execution costs in dark pools, although they do not explicitly measure the costs of adverse selection.

7 Concluding Remarks

In recent years, dark pools have become an important part of equity market structure. This paper provides a simple model of dark pool trading and their effects on price discovery and liquidity. I show that under natural conditions, the addition of a dark pool concentrates informed traders on the exchange and improves price discovery, at the cost of reducing exchange liquidity.

Besides price discovery and liquidity, there are a few additional aspects of dark pools that contribute to their controversy. One of these is information leakage. In practice, a dark pool may send an "indication of interest" (IOI), which contains selected order information such as the ticker, to potential counterparties in order to facilitate a match. In this sense, these dark pools are not completely dark. For example, Buti, Rindi, and Werner (2011b) considers a setting where selected traders are informed of the state of the dark pool. The Securities and Exchange Commission (2009) proposed to treat actionable IOIs—IOIs containing the symbol, size, side, and price of an order—as quotes, which must be disseminated to the broad market immediately.

Another consideration is fair access. In the United States, dark pools are not required to provide fair access unless the dark pool concerned reaches a 5% volume threshold. Whether investors suffer from the lack of fair access can depend on perspective. On the one hand, it seems plausible that the lack of fair access can reduce trading opportunities and the welfare of excluded traders. On the other hand, "some dark pools attempt to protect institutional trading interest by raising access barrier to the sell-side or certain hedge funds," observes SEC Deputy Director James Brigagliano.[25] For example, results from Boni, Brown, and Leach (2012) indicate that the exclusion of short-term traders in a dark pool (Liquidnet) improves the execution quality of institutional orders. Foster, Gervais, and Ramaswamy (2007) theoretically illustrate that setting a volume threshold in the dark pool—i.e. the dark pool executes orders only if trading interests on both sides of the market reach that threshold—can sometimes prevent impatient traders or informed traders from participating in the dark pool.

Finally, dark pools are opaque not only in their orders, but also in their trading mechanisms. For example, a Greenwich Associates survey of 64 active institutional users of dark pools reveals that, on many occasions, dark pools do not disclose sufficient information regarding the types of orders that are accepted, how orders interact with each other, how customers' orders are routed, what anti-gaming controls are in place, whether customer orders are exposed to proprietary trading flows, and at what price orders are matched (Bennett, Colon, Feng, and Litwin, 2010). The International Organization of Securities Commissions (2010) also observes that "[l]ack of information about the operations of dark pools and dark orders may result in market participants making uninformed decisions regarding whether or how to trade within a dark pool or using a dark order." Opaque operating mechanics of dark pools can make it more difficult for investors and regulators to evaluate the impact of dark pools on price discovery, liquidity, and market quality.

[25] "Keynote Speech to the National Organization of Investment Professionals," by James A. Brigagliano, April 19, 2010.

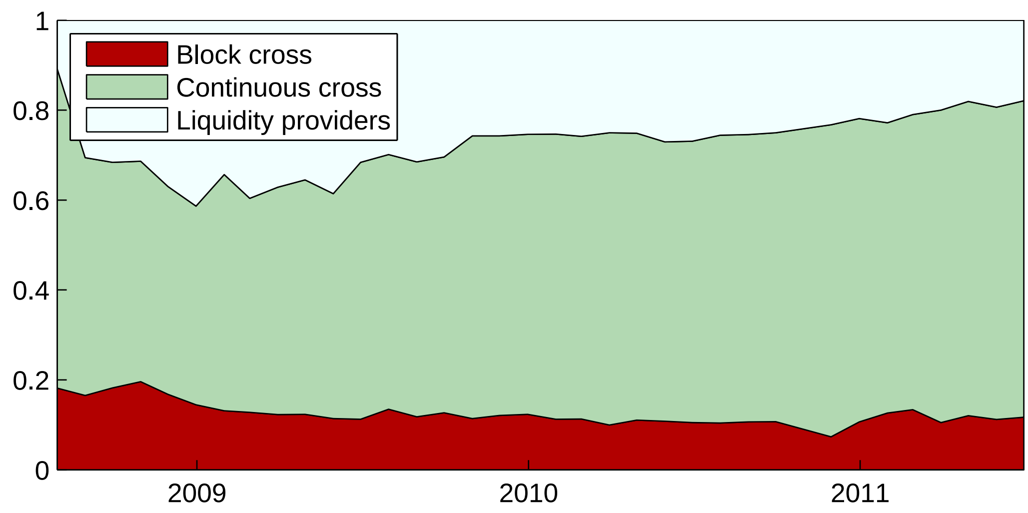
Appendix

A Institutional Features of Dark Pools

This appendix discusses additional institutional features of dark pools and nondisplayed liquidity that are not covered in Section 2.

Besides the three-way classification of dark pools discussed in Section 2, another classification is provided by Tabb Group (2011). They categorize dark pools into block-cross platforms, continuous-cross platforms, and liquidity-provider platforms. The main features of these three groups are summarized in the top panel of Table 2, and their respective market shares are plotted in Figure 8. As we can see, the market share of block-cross dark pools has declined from nearly 20% in 2008 to just above 10% in 2011. Continuous-cross dark pools have gained market share during the same period, from around 50% to around 70%. The market share of liquidity-providing dark pools increased to about 40% around 2009, but then declined to about 20% in mid 2011. Tabb Group's data, however, do not cover the entire universe of dark pools, and the components of each category can vary over time. For this reason, these statistics are noisy and should be interpreted with caution.

Figure 8: Market shares of three types of U.S. dark pools as fractions of total U.S. dark pool volume, estimated by Tabb Group. The three types are summarized in the top panel of Table 2.



Dark pools are also commonly classified by their crossing frequencies and by how they search for matching counterparties, as illustrated in the bottom panel of Table 2.

Table 2: Dark pools classification by trading mechanisms. The top panel shows the classification of Tabb Group. The bottom panel shows a classification by the crossing frequencies and the methods of finding counterparties.

Classification by Tabb Group

Types	Examples	Notes
Block cross	Liquidnet, BIDS, Instinet Cross	Similar to the first group of the top panel
Continuous cross	Credit Suisse Crossfinder, Goldman Sachs Sigma X, Barclays LX, Morgan Stanley MS Pool, LeveL, Deutsche Bank SuperX	Similar to the second group of the top panel; LeveL is owned by a consortium of broker-dealers
Liquidity provider	Getco and Knight	Same as the third group of the top panel

Classification by trading frequency and counterparty search

Types		Examples	Typical features
Scheduled		ITG POSIT Match, Instinet US Crossing	Cross at fixed clock times, with some randomization
Continuous	Matching	ITG POSIT Now, Instinet CBX, Direct Edge MidPoint Match	
	Advertized	Pipeline, POSIT Alert, Liquidnet	Electronic messages sent to potential matched counterparties
	Negotiated	Liquidnet	
	Internal	Credit Suisse Crossfinder, Goldman Sachs Sigma X, Knight Link, Getco	Owned by broker-dealers, run as nondisplayed limit order books or electronic market makers

Aside from mechanisms such as midpoint-matching and limit order books, advertisement is sometimes used to send selected information about orders resting in the dark pool to potential counterparties, in order to facilitate a match.

Characteristics that distinguish dark pools also include ownership structure and order size. Today, most dark pools are owned by broker-dealers (with or without proprietary order flows), whereas a small fraction is owned by consortiums of broker-dealers or exchanges. Order sizes can also vary substantially across dark pools. According to Rosenblatt Securities (2011), two block-size dark pools (Liquidnet and Pipeline) have an order size of around 50,000 shares, which is larger than that of Posit (around 6,000 shares per order) and much larger than those of other broker dark pools (about 300 shares per order). This sharp contrast in order sizes can be attributed to the use of algorithms that split "parent" orders into smaller "children" orders, as observed by the Securities and Exchange Commission (2010).

There are at least two reasons why high-quality data are lacking on dark pool trading in the United States. First, in the United States, dark pool trades are reported to "trade reporting facilities," or TRFs, which aggregate trades executed by all off-exchange venues—including dark pools, ECNs, and broker-dealer internalization—into a single category. Thus, it is generally not possible to assign a TRF trade to a specific off-exchange venue that executes the trade.[26] Second, dark pools often do not have their own identification numbers (MPID) for trade reporting. For example, a broker-dealer may report customer-to-customer trades in its dark pool *together with* the broker's own over-the-counter trades with institutions, all under the same MPID. Similarly, trades in an exchange-owned dark pool can be reported together with trades conducted on the exchange's open limit order book, all under the exchange's MPID. Because different trading mechanisms share the same MPID, knowing the MPID that executes a trade is insufficient to determine whether that trade occurred in a dark pool.[27]

Finally, there are two sources of nondisplayed liquidity that are usually not referred to as dark pools. One is broker-dealer internalization, by which a broker-dealer handles customer orders as a principal or an agent (Securities and Exchange Commission, 2010). A crude way of distinguishing dark pools from broker-dealer internalization is that the former are often marketplaces that allow direct customer-to-customer trades, whereas the latter typically involves broker-dealers as intermediaries.[28] The other source of

[26]The Securities and Exchange Commission (2009) has recently proposed a rule requiring that alternative trading systems (ATS), including dark pools, provide real-time disclosure of their identities on their trade reports.

[27]For example, Ye (2010) finds that only eight U.S. dark pools can be uniquely identified by MPIDs from their Rule 605 reports to the SEC. The majority of dark pools cannot.

[28]There are exceptions. For example, dark pools acting like electronic market makers (like Getco and Knight) also provide liquidity by trading on their own accounts. Nonetheless, they are highly automated systems and rely

nondisplayed liquidity is the use of hidden orders on exchanges. Examples include reserve ("iceberg") orders and pegged orders, which are limit orders that are partially or fully hidden from the public view.[29] For example, Nasdaq reports that more than 15% of its order flow is nondisplayed.[30] In particular, midpoint-pegged orders on exchanges are similar to dark pool orders waiting to be matched at the midpoint.

B Dynamic Trading with Stochastic Crossing

This appendix explicitly characterizes a family of dynamic equilibria in which informed traders do not participate in the dark pool. Different from Section 5, information and traders in this section arrive at Poisson times, which give rise to tractable stationary equilibria.

Time is continuous, $t \geq 0$, and the market opens at time 0. As before, an asset pays an uncertain dividend v that is $+\sigma$ or $-\sigma$ with equal probabilities. The time of the dividend payment is exponentially distributed with mean $1/\lambda_F$, for $\lambda_F > 0$. Two types of risk-neutral traders—liquidity traders and informed traders—have independent Poisson arrivals with respective mean arrival rates of λ_L and λ_I. (Traders are thus "discrete.") Each trader can buy or sell one unit of the asset. As in Section 5, I do not consider endogenous information acquisition here. Upon arrival, an informed trader observes v perfectly. Liquidity traders, who are not informed regarding the dividend, arrive with an unwanted position in the asset whose size is either $+1$ or -1, equally likely and independent of all else.

As before, a lit exchange and a dark pool operate in parallel. A competitive and risk-neutral market maker on the exchange continually posts bid and ask prices for one unit of the asset, as in Glosten and Milgrom (1985). Any order sent to the exchange is immediately executed at the bid or the ask, and trade information is immediately disseminated to everyone. By competitive pricing, the bid price at any time t is the conditional expected asset value given the arrival of a new sell order at time t and given all public information up to, but before, time t. The ask price is set likewise. The market maker also maintains a public "midpoint" price that is the conditional expected asset value given all public information up to but before time t. Once an

less on human intervention than, say, dealers arranging trades over the telephone.

[29] A reserve order consists of a displayed part, say 200 shares, and a hidden part, say 1,800 shares. Once the displayed part is executed, the same amount, taken from the hidden part, becomes displayed, until the entire order is executed or canceled. Pegged orders are often fully hidden. Typically, pegged orders and hidden portions of reserve orders have lower execution priority than displayed orders with the same limit price.

[30] See http://www.nasdaqtrader.com/Trader.aspx?id=DarkLiquidity

exchange order is executed, the market maker immediately updates her bid, ask, and midpoint prices.

The dark pool accepts orders continually, and an order sent to the dark pool is observable only by the order submitter. The dark pool executes orders at the midpoint price and at the event times of a Poisson process with intensity λ_C that is independent of all else. Allocation in the dark pool is pro-rata on the heavier side, as in Section 3. For analytical tractability, I assume that unmatched orders in the dark pool are immediately sent to the exchange market maker, who then executes these orders at the conditional expected asset value given all past public information and given the quantity and direction of unmatched orders from the dark pool.

As in Section 5, the conditional likelihood ratio of v at time t is

$$R_t = \frac{\mathbb{P}_t(v = +\sigma)}{\mathbb{P}_t(v = -\sigma)}, \tag{45}$$

where $\mathbb{P}_t$ denotes the market maker's conditional probability. By construction, $R_0 = 1$. The conditional expected asset value is, as in Section 5,

$$V(R_t) = \sigma(\mathbb{P}_t(v = +\sigma) - \mathbb{P}(v = -\sigma)) = \frac{R_t - 1}{R_t + 1}\,\sigma. \tag{46}$$

To calculate the bid and ask prices, I let λ_t be the time-t arrival intensity (conditional mean arrival rate) of traders of any type to the exchange, and let μ_t be the time-t conditional probability that an arriving exchange trader is informed. Then,

$$q_t = \mu_t + (1 - \mu_t)0.5 = 0.5 + 0.5\mu_t \tag{47}$$

is the probability that an exchange trader arriving at t is "correct," that is, buying if $v = +\sigma$ and selling if $v = -\sigma$. The likelihood ratio

$$z_t = \frac{q_t}{1 - q_t} \tag{48}$$

then represents the informativeness of a time-t exchange order.[31] For example, if a buy order hits the market maker's bid at time t, then Bayes' Rule implies that

$$R_t = \frac{\mathbb{P}_t(v = +\sigma \,|\, Q = 1)}{\mathbb{P}_t(v = -\sigma \,|\, Q = 1)} = \frac{\mathbb{P}_t(Q = 1 \,|\, v = +\sigma)}{\mathbb{P}_t(Q = 1 \,|\, v = -\sigma)} \cdot \frac{\mathbb{P}_{t-}(v = +\sigma)}{\mathbb{P}_{t-}(v = -\sigma)} = R_{t-} z_t, \tag{49}$$

[31] In the equilibria characterized in this section, the information content of a buy order is equal to that of a sell order, so there is no need to specify them separately.

where $\mathbb{P}_{t-}$ denotes the market maker's probability conditional on all exchange transactions up to but before time t, and where $R_{t-} \equiv \lim_{s\uparrow t} R_s$. Similarly, if an exchange sell order arrives at time t, then

$$R_t = \frac{\mathbb{P}_t(v = +\sigma \,|\, Q = -1)}{\mathbb{P}_t(v = -\sigma \,|\, Q = -1)} = \frac{R_{t-}}{z_t}. \tag{50}$$

To break even, the market maker quotes a time-t bid price of $V(R_t z_t^{-1})$ and a time-t ask price of $V(R_t z_t)$. Because $V(\,\cdot\,)$ is nonlinear, $V(R_t)$ is generally not identical to the bid-ask midpoint, $(V(R_t z_t) + V(R_t z_t^{-1}))/2$. Nonetheless, for simplicity I refer to $V(R_t)$ as the "midpoint" price.

Liquidity traders must hold collateral equal to the expected loss on their unwanted risky positions. With probability κ_j and independently of all else, an arriving liquidity trader incurs a cost of γ_j per unit of time for every unit of collateral support in her risky position, where $(\kappa_j)_{j=1}^J$ and $(\gamma_j)_{j=1}^J$ are commonly-known constants and satisfy

$$0 \le \gamma_1 < \gamma_2 < \cdots < \gamma_{J-1} < \gamma_J, \tag{51}$$

$$\sum_{j=1}^{J} \kappa_j = 1. \tag{52}$$

Before executing her order, a liquidity buyer of type j incurs a flow cost of

$$c_t^j = \gamma_j \mathbb{E}_t[\max(0, v - V(R_t))] = \gamma_j \frac{R_t}{R_t + 1} \cdot \left(1 - \frac{R_t - 1}{R_t + 1}\right)\sigma = \gamma_j \frac{2R_t}{(R_t + 1)^2}\,\sigma. \tag{53}$$

A liquidity seller of type j has the same flow cost c_t^j because

$$\gamma_j \mathbb{E}_t[\max(0, V(R_t) - v)] = \gamma_j \frac{1}{R_t + 1} \cdot \left(\frac{R_t - 1}{R_t + 1} - (-1)\right)\sigma = \gamma_j \frac{2R_t}{(R_t + 1)^2}\,\sigma. \tag{54}$$

By independent splitting of Poisson processes, the arrival intensities of type-j liquidity buyers and type-j liquidity sellers are both $0.5\kappa_j\lambda_L$.

Without loss of generality, we focus on the strategies of informed buyers and liquidity buyers, whose payoffs are denoted $W(R_t)$ and $X(R_t)$, respectively. For simplicity, I use $\mathbb{E}_t^i[\,\cdot\,]$ as a shorthand for $\mathbb{E}_t[\,\cdot \,|\, v = \sigma]$, where the superscript "i" stands for "informed." Because I look for stationary equilibria, the payoffs $W(R_t)$ and $X(R_t)$ depend on the public information R_t but not on time t.

Proposition 6. *For fixed integer $M \in \{0, 1, 2, \ldots, J\}$, define*

$$z_e = \frac{\lambda_I + 0.5 \sum_{i=M}^{J} \kappa_i \lambda_L}{0.5 \sum_{i=M}^{J} \kappa_i \lambda_L}. \tag{55}$$

Under the conditions

$$\lambda_C < \frac{\sum_{i=M}^{J} \kappa_j \lambda_L}{2\lambda_I} \lambda_F, \tag{56}$$

$$\gamma_j < (\lambda_C + \lambda_F) \frac{z_e - 1}{z_e}, \quad 1 \leq j < M, \tag{57}$$

$$\gamma_j > (\lambda_C + \lambda_F)(z_e - 1) + \left(\lambda_I + \sum_{i=M}^{J} \kappa_i \lambda_L \right) \frac{(z_e - 1)^3}{z_e(\sqrt{z_e} + 1)^2}, \quad M \leq j \leq J, \tag{58}$$

there exists an equilibrium in which:

1. *Informed traders trade on the exchange immediately upon arrival.*
2. *Type-j liquidity traders, $M \leq j \leq J$, trade immediately on the exchange upon arrival.*
3. *Type-j liquidity traders, $1 \leq j < M$, enter orders in the dark pool. If the dark pool has not crossed by the time that the dividend is paid, they cancel their dark pool orders and trade immediately on the exchange.*
4. *At time t, the market maker quotes a bid of $V(R_t z_e^{-1})$ and an ask of $V(R_t z_e)$. Moreover, immediately after a dark pool crossing, the market maker executes all outstanding orders at a price of $V(R_t)$. Immediately after the dividend v is paid, the market maker executes all outstanding orders at the cum-dividend price of v.*

Proof. See Appendix C. □

A key step in the equilibrium solution of Proposition 6 is that informed traders expect the exchange price to move against them over time, but liquidity traders expect the exchange spread to narrow over time. Thus, informed traders are relatively impatient, whereas liquidity traders are relatively patient. These different expectations of future prices, as formally stated in the following lemma, underlie the partial separation between informed traders and liquidity traders in the equilibria of Proposition 6.

Lemma 1. *Let Q be the direction of the next exchange order that arrives before the dividend payment, that is, $Q = 1$ denotes a buy order and $Q = -1$ denotes a sell order. Under the strategies stated in Proposition 6:*

- *The asset value is a martingale for liquidity traders and the public, in that*

$$V(R_t) = \mathbb{E}_t[V(R_t z_e^Q)]. \tag{59}$$

- *The exchange ask price is a submartingale for informed buyers, in that*

$$V(R_t z_e) < \mathbb{E}_t^i[V(R_t z_e^Q z_e)]. \tag{60}$$

- *The exchange ask price is a supermartingale for liquidity buyers, in that*

$$\mathbb{E}_t[V(R_t z_e^Q z_e)] = V(R_t z_e) - \frac{2R_t^2(z_e-1)^3}{(R_t+1)^2(R_t z_e+1)(R_t z_e^2+1)}\sigma. \tag{61}$$

Proof. See Appendix C. □

In Proposition 6, z_e reflects the degree of information asymmetry on the exchange because it is the ratio of the mean arrival rate of traders in the "correct" direction versus the mean arrival rate of traders in the "wrong" direction. Proposition 6 says that an informed trader trades immediately on the exchange if the crossing frequency λ_C of the dark pool is sufficiently low relative to the risk that her private information becomes stale. A liquidity trader sends her order to the dark pool if and only if her delay cost γ is sufficiently low compared to the potential price improvement obtained by trading at the market midpoint. Moreover, because the exchange order informativeness z_e is increasing in M, the more liquidity traders trade in the dark pool, the more informative are exchange orders. This property is a dynamic analogue of the two-period equilibrium of Section 3.

We now briefly discuss the comparative statics of the equilibria, based on the tightness of the incentive constraints (56)-(58). First, a higher crossing frequency λ_C tightens (56), suggesting that informed traders are more likely to participate in the dark pool if the crossing frequency is higher. On the other hand, a higher λ_C relaxes (57) but tightens (58), making the dark pool more attractive to liquidity traders. As long as λ_C is sufficiently low, informed traders avoid in the dark pool. Second, a higher arrival rate λ_F of information relaxes (56), suggesting that informed traders are less likely to trade in the dark pool if they face a higher risk of losing their information advantage. By contrast, a higher λ_F makes the dark pool more attractive to liquidity traders by shortening their expected waiting time, as in (57)-(58). Third, a higher delay cost γ makes the dark pool less attractive to liquidity traders, without affecting the incentives of informed traders.

C Proofs

C.1 Proof of Proposition 1

I define $\hat{\mu}_I : [0, \infty) \to [0, \bar{\mu}]$ by

$$\hat{\mu}_I(s) = \bar{\mu} F\left(\frac{(1-G(1))\mu_z}{\hat{\mu}_I(s) + (1-G(1))\mu_z} s\right). \tag{62}$$

Given the value σ of information, $\hat{\mu}_I(\sigma)$ is the unique "knife-edge" mass of informed traders with the property that all informed traders and a fraction $1 - G(1)$ of liquidity traders send orders to the exchange.

To prove the proposition, I show that a Case 1 equilibrium exists if

$$\bar{r} \leq 1 - \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1-G(1))\mu_z}, \tag{63}$$

and that a Case 2 equilibrium exists if and only if

$$\bar{r} > 1 - \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1-G(1))\mu_z}. \tag{64}$$

Then I show that the condition (63) is equivalent to $\sigma \leq \bar{\sigma}$ for some $\bar{\sigma}$, and that the condition (64) is equivalent to $\sigma > \bar{\sigma}$.

Clearly, $\beta < 1$; otherwise, the exchange spread would be zero and informed traders would deviate to trade on the exchange. Thus, in equilibrium either $\beta = 0$ or $0 < \beta < 1$.

We first look for an equilibrium in which $\beta = 0$. By (15), $\alpha_0 = 0$ and $\alpha_e = 1 - \alpha_d$. The indifference condition of the marginal liquidity trader is given by (16). For notational simplicity, we write the left-hand side of (16) as $-\tilde{X}_d(\alpha_d)$ and the right-hand side as $-\tilde{X}_e(\alpha_d)$. For each α_d, μ_I is uniquely determined by (17). We have

$$\begin{aligned}
&-\tilde{X}_d(0) = 0 < -\tilde{X}_e(0), \\
&-\tilde{X}_d(G(1)) = 1 - \bar{r} \geq \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1-G(1))\mu_z} = -\tilde{X}_e(G(1)),
\end{aligned}$$

where the second inequality follows from (63), (17), and (62). So there exists a solution $\alpha_d^* \in (0, G(1)]$ that satisfies (16).

Now we look for an equilibrium in which $\beta > 0$, that is, informed traders are indifferent between the exchange and the dark pool. What remains to be shown is that the incentive-compatibility conditions (18)-(20) have a solution. For simplicity, we write the left-hand side of (18) as $\tilde{W}_d(\beta)$ and the right-hand side of (18) as $\tilde{W}_e(\beta)$.

For each $\beta \geq 0$, μ_I is unique determined by (20) and is increasing in β. Under condition (64) and for each $\alpha_d > 0$,

$$\begin{aligned}
\tilde{W}_d(0) &= \bar{r} > 1 - \frac{\hat{\mu}_I(\sigma)}{\hat{\mu}_I(\sigma) + (1 - G(1))\mu_z} = \tilde{W}_e(0), \\
\tilde{W}_d(1) &= r^- < 1 = \tilde{W}_e(1),
\end{aligned}$$

where the first inequality follows from (63), (17), and (62). So there exists a solution $\beta^* \in (0,1)$ to (18), as a function of α_d. Because μ_I increases in β, we see that $\tilde{W}_d'(\beta) < 0$ and $\tilde{W}_e'(\beta) > 0$, holding α_d fixed. Thus, the solution β^* to (18) is unique for each α_d.

Moreover, (18) implies that in equilibrium r^- is bounded away from 0. So there exists some $r_0 > 0$ such that $r^- > r_0$. So for sufficiently small $\alpha_d > 0$,

$$G(1) - G\left(\frac{r^+ - r^-}{r^+ + r^-}\right) > G(1) - G\left(\frac{1 - r_0}{1 + r_0}\right) > \alpha_d.$$

So there exists a solution $\alpha_d^* \in (0, G(1)]$ to (19). The equilibria characterized by (18)-(20) thus exist. To show that (64) is necessary for the existence of equilibria in which $\beta > 0$, suppose for contradiction that (64) does not hold. Then, for all α_d and $\beta > 0$, $\tilde{W}_e(\beta) > \tilde{W}_e(0) \geq \tilde{W}_d(0) > \tilde{W}_d(\beta)$, which implies that all informed traders wish to deviate to the exchange, contradicting $\beta > 0$.

Finally, by (62), increasing the value σ of information raises the knife-edge mass $\hat{\mu}(\sigma)$ of informed traders, which in turn tightens the condition (63) under which informed traders avoid the dark pool. Thus, there exists some unique volatility threshold $\bar{\sigma}$ at which (63) holds with an equality. That is, the equilibrium in Case 1 exists if $\sigma \leq \bar{\sigma}$, and the equilibrium in Case 2 exists if $\sigma > \bar{\sigma}$.

C.2 Proof of Proposition 2

Because β, α_d, α_e, μ, S, r^+ and r^- are implicitly defined by differentiable functions in each case of Proposition 1, they are continuous and differentiable in σ in each of the two intervals $[0, \bar{\sigma}]$ and $(\bar{\sigma}, \infty)$. At the volatility threshold $\sigma = \bar{\sigma}$, differentiability refers to right-differentiability in Case 1 of Proposition 1, and left-differentiability in Case 2.

Have a dark pool and $\sigma \le \bar{\sigma}$

For $\sigma \le \bar{\sigma}$, $\beta = 0$. Total differentiation of (16)-(17) with respect to σ yields

$$\underbrace{\left[\frac{dG^{-1}(\alpha_d)}{d\alpha_d}(1-\bar{r}) - \frac{\partial(S/\sigma)}{\partial\alpha_d}\right]}_{>0}\frac{d\alpha_d}{d\sigma} - \underbrace{\frac{\partial(S/\sigma)}{\partial\mu_I}}_{>0}\frac{d\mu_I}{d\sigma} = 0, \tag{65}$$

$$\underbrace{\left[1 - \bar{\mu}F'(\sigma-S)\frac{\partial\,(\sigma-S)}{\partial\mu_I}\right]}_{>0}\frac{d\mu_I}{d\sigma} = \underbrace{\bar{\mu}F'(\sigma-S)\frac{\partial(\sigma-S)}{\partial\alpha_d}}_{<0}\frac{d\alpha_d}{d\sigma} + \underbrace{\bar{\mu}F'(\sigma-S)\left(1-\frac{S}{\sigma}\right)}_{>0}, \tag{66}$$

where the first term of (65) is positive because of equilibrium selection. If $d\alpha_d/d\sigma \le 0$ at, say, some σ_0, then (66) implies that $d\mu_I/d\sigma > 0$ at σ_0. But then (65) cannot hold. Thus, $d\alpha_d/d\sigma > 0$, $d\mu_I/d\sigma > 0$, and $d(S/\sigma)/d\sigma > 0$, by (16).

Have a dark pool and $\sigma > \bar{\sigma}$

Now suppose that $\sigma > \bar{\sigma}$. I denote by $r^{+'}$ and $r^{-'}$ the derivatives of r^+ and r^- with respect to $\beta\mu_I/\alpha_d$. We have $r^{+'} > 0$ and $r^{-'} < 0$. Total differentiation of (18)-(20) with respect to σ yields

$$\underbrace{\left(r^{-'}\frac{1}{\alpha_d} - \frac{\partial(1-S/\sigma)}{\partial(\beta\mu_I)}\right)}_{<0}\frac{d(\beta\mu_I)}{d\sigma} = \underbrace{\frac{\partial(1-S/\sigma)}{\partial\mu_I}}_{<0}\frac{d\mu_I}{d\sigma} + \underbrace{r^{-'}\frac{\beta\mu_I}{\alpha_d^2}}_{<0}\frac{d\alpha_d}{d\sigma}, \tag{67}$$

$$\underbrace{\left[1 - G'\left(\frac{r^+-r^-}{r^++r^-}\right)\frac{2(r^{+'}r^- - r^{-'}r^+)}{(r^++r^-)^2}\frac{\beta\mu_I}{\alpha_d^2}\right]}_{>0}\frac{d\alpha_d}{d\sigma} = -\underbrace{G'\left(\frac{r^+-r^-}{r^++r^-}\right)\frac{2(r^{+'}r^- - r^{-'}r^+)}{(r^++r^-)^2}\frac{1}{\alpha_d}}_{>0}\frac{d(\beta\mu_I)}{d\sigma}, \tag{68}$$

$$\underbrace{\left[1 - \bar{\mu}F'(\sigma-S)\sigma\frac{\partial(1-S/\sigma)}{\partial\mu_I}\right]}_{>0}\frac{d\mu_I}{d\sigma} = \underbrace{\bar{\mu}F'(\sigma-S)\sigma\frac{\partial(1-S/\sigma)}{\partial(\beta\mu_I)}}_{>0}\frac{d(\beta\mu_I)}{d\sigma} + \underbrace{\bar{\mu}F'(\sigma-S)\left(1-\frac{S}{\sigma}\right)}_{>0}, \tag{69}$$

where the first term of (68) is positive because of equilibrium selection.

We can show that $d\alpha_d/d\sigma$ cannot switch signs in $[\bar{\sigma}, \infty)$. To see why, suppose otherwise, and $d\alpha_d/d\sigma$ switches signs at some σ_0. By continuity, at σ_0, $d\alpha_d/d\sigma = 0$. But (68) and (67) imply that $d(\beta\mu_I)/d\sigma = 0 = d\mu_I/d\sigma$ at σ_0 as well, which contradicts (69). Thus, $d\alpha_d/d\sigma$ cannot switch signs in $[\bar{\sigma}, \infty)$; nor can it be zero.

At $\sigma = \bar{\sigma}$, $\beta = 0$ and $d\beta/d\sigma \geq 0$. Then, by (68),

$$\left.\frac{d(\beta\mu_I)}{d\sigma}\right|_{\sigma=\bar{\sigma}} = \mu_I \left.\frac{d\beta}{d\sigma}\right|_{\sigma=\bar{\sigma}} \geq 0 \Longrightarrow \left.\frac{d\alpha_d}{d\sigma}\right|_{\sigma=\bar{\sigma}} \leq 0.$$

Because $d\alpha_d/d\sigma$ cannot be zero, it must be strictly negative for all $\sigma \in [\bar{\sigma}, \infty)$. By (68)-(69), for all $\sigma \in [\bar{\sigma}, \infty)$, $\beta\mu_I$ and μ_I are both strictly increasing in σ. Then, (18) implies that

$$\frac{d(S/\sigma)}{d\sigma} = -\frac{dr^-}{d\sigma} = -r^{-\prime}\frac{d}{d\sigma}\left(\frac{\beta\mu_I}{\alpha_d}\right) > 0.$$

The spread itself, $S = \sigma \cdot (S/\sigma)$, obviously increases in σ as well. Finally,

$$\frac{dr^+}{d\sigma} = r^{+\prime}\frac{d}{d\sigma}\left(\frac{\beta\mu_I}{\alpha_d}\right) > 0, \quad \frac{dr^-}{d\sigma} = r^{-\prime}\frac{d}{d\sigma}\left(\frac{\beta\mu_I}{\alpha_d}\right) < 0.$$

No dark pool

The comparative statics for Corollary 1 are similar to that for the first case of Proposition 1 and are omitted.

C.3 Proof of Proposition 3

Have a dark pool and $\sigma \leq \bar{\sigma}$

For $\sigma \leq \bar{\sigma}$, adding a dark pool is equivalent to increasing $\bar{r}$. Total differentiation of (16)-(17) with respect to $\bar{r}$ yields

$$\underbrace{\left[(1-\bar{r})\frac{\partial G^{-1}(\alpha_d)}{\partial\alpha_d} - \frac{\partial(S/\sigma)}{\partial\alpha_d}\right]}_{>0}\frac{d\alpha_d}{d\bar{r}} = G^{-1}(\alpha_d) + \underbrace{\frac{\partial(S/\sigma)}{\partial\mu_I}}_{>0}\frac{d\mu_I}{d\bar{r}}, \tag{70}$$

$$\underbrace{\left[1 - \bar{\mu}F'(\sigma - S)\frac{\partial}{\partial\mu_I}(\sigma - S)\right]}_{>0}\frac{d\mu_I}{d\bar{r}} = \underbrace{\bar{\mu}F'(\sigma - S)\frac{\partial(\sigma - S)}{\partial\alpha_d}}_{<0}\frac{d\alpha_d}{d\bar{r}}, \tag{71}$$

where the first term on the left-hand side of (70) is positive because of the equilibrium selection. If $d\alpha_d/d\bar{r} \leq 0$ at any σ_0, then (71) implies that $d\mu_I/d\bar{r} \geq 0$ at σ_0. But that contradicts (70). Thus, $d\alpha_d/d\bar{r} > 0$ and $d\mu_I/d\bar{r} < 0$. Adding a dark pool, which is equivalent to an increase in $\bar{r}$, raises α_d and reduces $\alpha_e = 1-\alpha_d$. The total participation rate of liquidity traders in either the dark pool or the exchange is $\alpha_d + \alpha_e = 1$, higher than a market without a dark pool. Moreover, by (17), a lower μ_I implies a wider spread S on the exchange.

Have a dark pool and $\sigma > \bar{\sigma}$

Now suppose that $\sigma > \bar{\sigma}$. In a market with a dark pool, $\alpha_e = 1 - G(1)$, a constant. Substituting it into (21) and we have

$$\frac{\mu_I}{\mu_I + (1 - G(1))\mu_z} < 1.$$

So the equilibrium α_e without a dark pool resides in the interval $(1 - G(1), 1)$. That is, adding a dark pool reduces α_e.

Moreover, adding a dark pool increases the exchange spread if and only if α_e in the equilibrium of Corollary 1 is larger than $(1-G(1))/(1-\beta)$, where $\beta > 0$ is determined in Proposition 1. By the equilibrium selection rule and by (18),

$$\alpha_e > \frac{1 - G(1)}{1 - \beta} \Longleftrightarrow G^{-1}\left(1 - \frac{1 - G(1)}{1 - \beta}\right) > \frac{\mu_I}{\mu_I + \mu_z(1 - G(1))/(1 - \beta)} = 1 - r^-, \tag{72}$$

where the μ_I is given by

$$\mu_I = \bar{\mu} F\left(\frac{(1 - G(1))\mu_z}{(1 - \beta)\mu_I + (1 - G(1))\mu_z}\right).$$

We rearrange (72) and obtain

$$\beta < \frac{G(1) - G(1 - r^-)}{1 - G(1 - r^-)}.$$

On the other hand, because the left-hand side of (18) is decreasing in β and the right-hand side is increasing in β, the above condition is equivalent to (23).

As $F(c) \to 1$ for all $c > 0$, (20) implies that $\mu_I \to \bar{\mu}$, a constant. Holding $\mu_I = \bar{\mu}$ fixed, we now show that if $G''(1 - r^-) \leq 0$, then (23) holds for all $r^- \in [0, \bar{r}]$. At $r^- = \bar{r}$, we have $\sigma = \bar{\sigma}$ and (23) holds by the definition of $\bar{\sigma}$. At $r^- = 0$, (23) also holds trivially. Take the first and second derivatives of the right-hand side of (23) with respect to r^- and we obtain

$$\begin{aligned}
\frac{d[rhs(23)]}{dr^-} &= \frac{\bar{\mu}\mu_z G'(1 - r^-)}{[\bar{\mu} + (1 - G(1 - r^-))\mu_z]^2} > 0, \\
\frac{d^2[rhs(23)]}{d(r^-)^2} &= \bar{\mu}\mu_z \frac{G''(1 - r^-)[\bar{\mu} + (1 - G(1 - r^-))\mu_z] - 2\mu_z[G'(1 - r^-)]^2}{[\bar{\mu} + (1 - G(1 - r^-))\mu_z]^3} < 0.
\end{aligned}$$

Thus, the right-hand side of (23) is concave and (23) holds for all $r^- \in [0, \bar{r}]$.

C.4 Proof of Proposition 4

I prove this proposition in three steps. First, I calculate the execution price in the dark pool and the optimal limit prices chosen by liquidity traders. Second, I derive incentive-compatibility conditions under which informed traders choose not to participate in the dark pool. Finally, I prove the comparative statics and conditions under which the equilibrium exists.

Step 1: Price p^* of execution and optimal prices of limit orders

I let $y^+ : [-xS, xS] \to [0, \infty)$ be the aggregate *downward-sloping* demand schedule of liquidity buyers in the dark pool, and let $y^- : [-xS, xS] \to [0, \infty)$ be the aggregate *upward-sloping* supply schedule of liquidity sellers. For each p, $y^+(p)$ is the total mass of limit buy orders that have a limit price of p or higher, and $y^-(p)$ is the total mass of limit sell orders that have a limit price of p or lower. Because the dark pool crosses orders by price priority, its execution price p^* is

$$p^* = \begin{cases} xS, & \text{if } y^+(p) > y^-(p) \text{ for all } p \in [-xS, xS]. \\ -xS, & \text{if } y^+(p) < y^-(p) \text{ for all } p \in [-xS, xS]. \\ \{p : y^+(p) = y^-(p)\}, & \text{otherwise.} \end{cases} \tag{73}$$

I proceed under the conjecture that the set $\{p : y^+(p) = y^-(p)\}$ contains at most one element, in which case p^* of (73) is uniquely well-defined. I later verify this conjecture. Once p^* is determined, buy orders with limit prices above or equal to p^* are matched, at the price of p^*, with sell orders whose prices are at most p^*. If there is a positive mass of buy or sell orders at the price p^*, then traders setting the limit price p^* are rationed pro-rata, as before.

I now derive the optimal limit prices of liquidity traders in the dark pool, under the conjecture that the probability distribution of p^* has no atom in $(-xS, xS)$. This no-atom conjecture, verified later, implies that a liquidity trader quoting a price of $p \in (-xS, xS)$ has her order filled with certainty (i.e. is not rationed) if $p^* = p$. Thus, a liquidity buyer who has a delay cost of $c \in [0, xS)$ and quotes a price of p in the dark pool has the expected payoff (negative cost)

$$\begin{aligned} X_d(p; c) &= -\mathbb{E}\left[\mathbb{I}_{\{p \geq p^*\}} p^* + \mathbb{I}_{\{p < p^*\}} c\right] \\ &= -c - \int_{-xS}^{p} (p^* - c)\, dH(p^*), \end{aligned} \tag{74}$$

where $\mathbb{I}(\,\cdot\,)$ is the indicator function and $H(p^*)$ is the cumulative distribution function of p^*. Because there is no adverse selection in the dark pool, the execution cost for this liquidity buyer is either the payment p^* or the delay cost c. Conjecturing that $H(p^*)$ is differentiable with $H'(p^*) > 0$ for $p^* \in (-xS, xS)$, properties that are also verified later, we obtain

$$\frac{dX_d(p;c)}{dp} = -(p-c)H'(p). \tag{75}$$

Because (75) shows that the sensitivity of expected payoff to the limit price p is positive for $p < c$ and negative for $p > c$, the optimal limit price for the liquidity buyer is her delay cost c. Symmetrically, the optimal limit price for a liquidity seller with a delay cost of $c \in [0, xS)$ is $-c$. This "truth-telling" strategy is also ex-post optimal, in that no one wishes to deviate even after observing the execution price. The first-order condition (75) also implies that xS is the highest limit price in the dark pool, and that $-xS$ is the lowest limit price.[32]

Let $y(p)$ be the downward-sloping demand schedule in the dark pool if $Z^+ = 1$. Because a limit price $p \in [0, xS)$ is submitted by the liquidity buyer with the delay cost p,

$$y(p) = \alpha_d - G\left(\frac{\max(0,p)}{\sigma}\right), \quad -xS < p < xS. \tag{76}$$

By symmetry, the liquidity buyers' demand schedule and the liquidity sellers' supply schedule in the dark pool are, respectively,

$$y^+(p) = Z^+ y(p), \tag{77}$$

$$y^-(p) = Z^- y(-p). \tag{78}$$

Because the equation $y^+(p) = y^-(p)$ has at most one root, we have verified our earlier conjecture that the dark pool execution price p^* is uniquely well-defined.

Given $y(p)$, the execution price p^* in the dark pool is

$$p^* = \begin{cases} +xS, & \text{if } [\alpha_d - G\left(\frac{xS}{\sigma}\right)]Z^+ \geq \alpha_d Z^-, \\ +\sigma G^{-1}\left[\alpha_d\left(1 - \frac{Z^-}{Z^+}\right)\right], & \text{if } \left[\alpha_d - G\left(\frac{xS}{\sigma}\right)\right] Z^+ < \alpha_d Z^- \leq \alpha_d Z^+, \\ -\sigma G^{-1}\left[\alpha_d\left(1 - \frac{Z^+}{Z^-}\right)\right], & \text{if } \left[\alpha_d - G\left(\frac{xS}{\sigma}\right)\right] Z^- < \alpha_d Z^+ \leq \alpha_d Z^-, \\ -xS, & \text{if } [\alpha_d - G\left(\frac{xS}{\sigma}\right)]Z^- \geq \alpha_d Z^+. \end{cases} \tag{79}$$

[32] If the maximum limit price were lower, say $p_0 < xS$, then a liquidity buyer with a delay cost of $p_0 + \epsilon$ for some small $\epsilon > 0$ would deviate to the dark pool and quote $p_0 + \epsilon$. This deviating buyer has an execution probability of 1 and pays at most $p_0 + \epsilon < xS \leq S$, which is better than execution on the exchange. The argument for the lowest limit price is symmetric.

Because the total trading interest Z^+ of liquidity buyer and the total trading interest Z^- of liquidity sellers are identically distributed, the dark pool execution price p^* has a mean of zero. By the differentiability of G and of the distribution function of Z^-/Z^+, $H(p^*)$ is continuous, differentiable, and strictly increasing on $(-xS, xS)$, as conjectured earlier.

Step 2: Incentive conditions for participation

What remains to be shown are the incentive-compatibility conditions of liquidity traders who set the limit price xS or $-xS$ in the dark pool, as well as the incentive-compatibility condition of informed traders, who avoid the dark pool. A liquidity buyer quoting the limit price xS in the dark pool has an execution probability of

$$\bar{r}_x = \mathbb{E}\left[\min\left(1, \frac{\alpha_d Z^-}{(\alpha_d - G(xS/\sigma))Z^+}\right)\right], \tag{80}$$

and an expected payoff, given the delay cost c, of

$$X_d(xS; c) = -(1 - \bar{r}_x)(c - xS). \tag{81}$$

This expected payoff calculation follows from the fact that $\mathbb{E}(p^*) = 0$ and the fact that failing to cross in the dark pool incurs a delay cost of c but saves the payment xS.

Because informed traders avoid the dark pool with probability 1 in the conjectured equilibrium, an informed buyer who *deviates* to the dark pool also has the crossing probability $\bar{r}_x$. Moreover, in order to get the highest priority, this deviating informed trader sets the highest limit price xS. Her expected profit in the dark pool is thus

$$W_d = \sigma - (1 - \bar{r}_x)(\sigma - xS). \tag{82}$$

As before, for any delay cost $c \leq \sigma$,

$$W_d - X_d(xS; c) = \sigma\bar{r}_x + c(1 - \bar{r}_x) \leq \sigma = W_e - X_e. \tag{83}$$

That is, an informed buyer behaves in the same way as does a liquidity buyer who has a delay cost of σ. If informed traders do not participate in the dark pool, an equilibrium is determined by a marginal liquidity trader who is indifferent between the dark pool and the exchange. Given α_d, this liquidity trader has a delay cost of $G^{-1}(\alpha_d)\sigma$. So we must have $X_d(xS; G^{-1}(\alpha_d)) = -S$, or (36). Thus, (36) and (37) characterize an equilibrium. We now look for conditions under which, in equilibrium, $\alpha_d^* \leq G(1)$. In this equilibrium, $\beta = 0$.

Step 3: Comparative statics and conditions for the existence of equilibria

I now calculate the comparative statics, assuming the existence of an equilibrium, and then show conditions under which the stated equilibrium exists. Total differentiation of (36) and (37) with respect to σ yields

$$\underbrace{\left(\frac{\partial[lhs(36)]}{\partial\alpha_d} - \frac{\partial[rhs(36)]}{\partial\alpha_d}\right)}_{>0}\frac{d\alpha_d}{d\sigma} + \underbrace{\left(\frac{\partial[lhs(36)]}{\partial\mu_I} - \frac{\partial[rhs(36)]}{\partial\mu_I}\right)}_{<0}\frac{d\mu_I}{d\sigma} = 0, \tag{84}$$

$$\underbrace{\left(1 - \frac{\partial[rhs(37)]}{\partial\mu_I}\right)}_{>0}\frac{d\mu_I}{d\sigma} = \underbrace{\frac{\partial[rhs(37)]}{\partial\alpha_d}}_{<0}\frac{d\alpha_d}{d\sigma} + \underbrace{\bar{\mu}F'(\sigma - S)\left(1 - \frac{S}{\sigma}\right)}_{>0}. \tag{85}$$

As before, if $d\alpha_d/d\sigma \le 0$ at some σ_0, then (84) implies that $d\mu_I/d\sigma \le 0$ at σ_0 as well. But this contradicts (85). Thus, the comparative statics with respect to σ follow. And given the equilibrium, the dark pool execution price p^* and the optimal limit prices follow from calculations done in Step 1 of the proof.

Now I characterize the condition for the existence of an equilibrium and the threshold volatility $\bar{\sigma}(x)$. For $x \in [0, 1]$, I define $\bar{K}(x)$ implicitly by

$$(1 - x\bar{K}(x))\left\{1 - \mathbb{E}\left[\min\left(1, \frac{G(1)Z^-}{[G(1) - G(x\bar{K}(x))]Z^+}\right)\right]\right\} = \bar{K}(x). \tag{86}$$

This $\bar{K}(x)$ is uniquely well-defined because the left-hand side of (86) is decreasing in $\bar{K}(x)$ and the right-hand side is strictly increasing in $\bar{K}(x)$. Moreover, total differentiation of (86) with respect to x yields

$$\underbrace{\left(\frac{\partial[lhs(86)]}{\partial\bar{K}(x)} - 1\right)}_{<0}\bar{K}'(x) + \underbrace{\frac{\partial[lhs(86)]}{\partial x}}_{<0} = 0.$$

So we have $\bar{K}'(x) < 0$.

On the other hand, given $\bar{K}(x)$, I define $\mu_I^*(x)$ by

$$\frac{\mu_I^*(x)}{\mu_I^*(x) + (1 - G(1))\mu_z} = \bar{K}(x),$$

and define $\bar{\sigma}(x)$ by

$$\mu_I^*(x) = \bar{\mu} F\left(\frac{(1-G(1))\mu_z}{\mu_I^*(x) + (1-G(1))\mu_z}\,\bar{\sigma}(x)\right).$$

Because $\mu_I^*(x)$ is strictly increasing in $\bar{K}(x)$ and because $\bar{\sigma}(x)$ is strictly increasing in $\mu_I^*(x)$, $\bar{\sigma}(x)$ is strictly increasing in $\bar{K}(x)$. Because $\bar{K}'(x) < 0$, $\bar{\sigma}'(x) < 0$.

What remains to be shown is that, for $\sigma \leq \bar{\sigma}(x)$, an equilibrium characterized by Proposition 4 exists. Clearly, once α_d is determined, μ_I is uniquely determined by (37). For sufficiently small α_d, the left-hand side of (36) is negative, whereas the right-hand side is strictly positive. For $\alpha_d = G(1)$, (37) implies that

$$\mu_I = \bar{\mu} F\left(\frac{(1-G(1))\mu_z}{\mu_I + (1-G(1))\mu_z}\,\sigma\right),$$

which is no larger than $\mu_I^*(x)$. Thus,

$$K \equiv \frac{\mu_I}{\mu_I + (1-G(1))\mu_z} = \frac{S}{\sigma}\bigg|_{\alpha_d = G(1)} \leq \bar{K}(x),$$

and, by the definition of $\bar{K}(x)$,

$$(1-xK)\left\{1 - \mathbb{E}\left[\min\left(1, \frac{G(1)Z^-}{[G(1)-G(xK)]Z^+}\right)\right]\right\} > K.$$

That is, at $\alpha_d = G(1)$, the left-hand side of (36) is weakly higher than the right-hand side. Therefore, there exists a solution $\alpha_d^* \in (0, G(1))$ to (36), and an equilibrium exists.

C.5 Proof of Proposition 5

Suppose that $W_d(R_t, t; \gamma) \geq \mathbb{E}_t[W(R_{t+1}, t+1; \gamma)]$ in an equilibrium. I denote by $\hat{X}(R_t, t; \gamma)$ the "auxiliary payoff" of a type-γ liquidity buyer who "imitates" the strategy of a type-γ informed buyer. That is, the imitating buyer behaves as if $v = +\sigma$. Clearly, such imitation is suboptimal for the liquidity buyer, so $X(R_t, t; \gamma) \geq \hat{X}(R_t, t; \gamma)$ for all R_t, t, and γ. For notional simplicity, in the calculations below I suppress the cost type γ and likelihood ratio R_t as function arguments.

Suppose that the informed buyer enters an order in the dark pool at time t. The imitating liquidity buyer does the same, by construction. Let $\hat{X}_d^+$ be the dark pool payoff of the imitating buyer conditional on $v = +\sigma$, and let $\hat{X}_d^-$ be the dark pool payoff of the imitating buyer conditional on $v = -\sigma$. I define $\hat{X}_e^+$ and $\hat{X}_e^-$ similarly. Then, we have

$$\begin{aligned}\hat{X}_d(t) - W_d(t) &= \frac{R_t}{R_t+1}\hat{X}_d^+(t) + \frac{1}{R_t+1}\hat{X}_d^-(t) - W_d(t)\\ &= \frac{1}{R_t+1}\left(\hat{X}_d^-(t) - \hat{X}_d^+(t)\right),\end{aligned}$$

where the last equality follows from the fact that, conditional on the true dividend, the expected payoff W_d of the informed buyer is the same as the payoff $\hat{X}_d^+$ of the imitating liquidity buyer. If we can show that $\hat{X}_d^+(t) - \hat{X}_d^-(t) \le 2\sigma$, then

$$X_d(t) - W_d(t) \ge \hat{X}_d(t) - W_d(t) \ge -\frac{2\sigma}{R_t+1} = X_e(R_t,t) - W_e(R_t,t).$$

Now I prove that $\hat{X}_d^+(t) - \hat{X}_d^-(t) \le 2\sigma$. I denote by $\mathcal{C}$ the event that the imitating buyer's order is crossed in the dark pool, and let

$$\begin{aligned}k_t^+ &\equiv \mathbb{P}_t[\mathcal{C} \,|\, v = +\sigma],\\ k_t^- &\equiv \mathbb{P}_t[\mathcal{C} \,|\, v = -\sigma],\end{aligned}$$

be the crossing probabilities of the imitating buyer in the dark pool in period t, conditional on $v = +\sigma$ and $v = -\sigma$, respectively. Then, we have

$$\begin{aligned}\hat{X}_d^+(t) &= k_t^+[+\sigma - \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = +\sigma)] + (1-k_t^+)\mathbb{E}_t(\hat{X}^+(t+1)),\\ \hat{X}_d^-(t) &= k_t^-[-\sigma - \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = -\sigma)] + (1-k_t^-)\mathbb{E}_t(\hat{X}^-(t+1)),\end{aligned}$$

where p^* is the execution price in the dark pool in period t, and $\hat{X}^+$ and $\hat{X}^-$ are the imitating buyer's payoffs conditional on $v = +\sigma$ and $v = -\sigma$, respectively. Informed buyers have either a zero or positive mass in the dark pool in period t, so

$$\begin{aligned}&k_t^+ \le k_t^-,\\ &\mathbb{E}_t(p^* \,|\, \mathcal{C}, v = +\sigma) \ge \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = -\sigma).\end{aligned}$$

Because $W_d(t) \ge \mathbb{E}_t[W(t+1)]$, $\hat{X}_d^+(t) \ge \mathbb{E}_t(\hat{X}^+(t+1))$. Thus,

$$\begin{aligned}\hat{X}_d^+(t) - \mathbb{E}_t(\hat{X}^+(t+1)) &= k_t^+[\sigma - \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = +\sigma) - \mathbb{E}_t(\hat{X}^+(t+1))]\\ &\le k_t^-[\sigma - \mathbb{E}_t(p^* \,|\, \mathcal{C}, v = -\sigma) - \mathbb{E}_t(\hat{X}^+(t+1))],\end{aligned}$$

which implies that

$$\hat{X}_d^+(t) - \hat{X}_d^-(t) \leq 2\sigma k_t^- + (1 - k_t^-)\mathbb{E}_t[\hat{X}^+(t+1) - \hat{X}^-(t+1)]. \tag{87}$$

I now prove that $\hat{X}_d^+(t) - \hat{X}_d^-(t) \leq 2\sigma$ and that $\hat{X}^+(t) - \hat{X}^-(t) \leq 2\sigma$ by induction. For all $t < T$, $X_e^+(t) - X_e^-(t) = 2\sigma$. Because v is revealed in period T, $\hat{X}^+(T) = \hat{X}^-(T) = 0$. By (87), $\hat{X}_d^+(T-1) - \hat{X}_d^-(T-1) \leq 2\sigma$. Because the venue choice of the imitating liquidity buyer does not depend on realizations of v,

$$\begin{aligned}
&X^+(T-1) - X^-(T-1) \\
&= \max\left[X_e^+(T-1) - X_e^-(T-1), X_d^+(T-1) - X_d^-(T-1), \mathbb{E}_{T-1}(X^+(T) - X^-(T))\right] \leq 2\sigma.
\end{aligned}$$

For the induction step, suppose that $\hat{X}^+(t+1) - \hat{X}^-(t+1) \leq 2\sigma$. Then, (87) implies that $X_d^+(t) - X_d^-(t) \leq 2\sigma$. Thus,

$$X^+(t) - X^-(t) = \max\left[X_e^+(t) - X_e^-(t), X_d^+(t) - X_d^-(t), \mathbb{E}_t(X^+(t+1) - X^-(t+1))\right] \leq 2\sigma,$$

which completes the proof.

C.6 Proof of Lemma 1

Given the public information R_t, the probability that $v = +\sigma$ is $R_t/(R_t+1)$. Let q be implicitly defined by $z_e = q/(1-q)$. That is, q is the probability that an arriving exchange order is in the same direction as informed orders. Under a liquidity trader's belief, the probability that the next exchange order is a buy order is

$$\frac{R_t}{R_t+1}q + \frac{1}{R_t+1}(1-q) = (1-q)\frac{R_t z_e + 1}{R_t+1}.$$

Similarly, the probability that the next exchange order is a sell order is

$$\frac{R_t}{R_t+1}(1-q) + \frac{1}{R_t+1}q = q\frac{R_t z_e^{-1} + 1}{R_t+1}.$$

We can verify that identity

$$
\begin{aligned}
&(1-q)\frac{R_t z_e+1}{R_t+1}V(R_t z_e)+q\frac{R_t z_e^{-1}+1}{R_t+1}V(R_t z_e^{-1})\\
=&(1-q)\frac{R_t z_e+1}{R_t+1}\left(1-\frac{2}{R_t z_e+1}\right)\sigma+q\frac{R_t z_e^{-1}+1}{R_t+1}\left(1-\frac{2}{R_t z_e^{-1}+1}\right)\sigma\\
=&\left(1-\frac{2}{R_t+1}\right)\sigma,
\end{aligned}
\tag{88}
$$

which implies $\mathbb{E}_t[V(R_t z_e^Q)] = V(R_t)$, that is, the expected midpoint price is a martingale for liquidity traders.

Note that the identity (88) holds for all R_t, including the informed trader's likelihood ratio ∞. Using (88) again, we have that

$$
\begin{aligned}
V(R_t z_e) &= \left(1-\frac{2}{R_t z_e+1}\right)\sigma\\
&= (1-q)\frac{R_t z_e z_e+1}{R_t z_e+1}\left(1-\frac{2}{R_t z_e z_e+1}\right)\sigma+q\frac{R_t z_e z_e^{-1}+1}{R_t z_e+1}\left(1-\frac{2}{R_t z_e z_e^{-1}+1}\right)\sigma\\
&< (1-q)z_e\left(1-\frac{2}{R_t z_e z_e+1}\right)\sigma+q z_e^{-1}\left(1-\frac{2}{R_t z_e z_e^{-1}+1}\right)\sigma\\
&= \mathbb{E}_t^i[V(R_t z_e^Q z_e)].
\end{aligned}
$$

That is, the exchange ask price is a submartingale for informed buyers.

Finally, direct calculation gives

$$
\begin{aligned}
\mathbb{E}_t[V(R_t z_e^Q z_e)] &= V(R_t z_e) - \frac{2R_t^2(z_e-1)^3}{(R_t+1)^2(R_t z_e+1)(R_t z_e^2+1)}\sigma,\\
\mathbb{E}_t[V(R_t z_e^Q z_e^{-1})] &= V(R_t z_e^{-1}) + \frac{2R_t^2(1-z_e^{-1})^3}{(R_t+1)^2(R_t z_e^{-1}+1)(R_t z_e^{-2}+1)}\sigma.
\end{aligned}
$$

That is, for liquidity traders, the exchange ask price is a supermartingale and the exchange bid price is a submartingale.

C.7 Proof of Proposition 6

I prove Proposition 6 by direct verification. The quoting strategy of the market maker simply follows from risk neutrality and zero profit.

Under the proposed equilibrium strategy, the total arrival intensity of exchange orders is $\lambda_t = \lambda_I + \sum_{i=M}^{J} \kappa_i \lambda_L$. The Hamilton-Jacobi-Bellman (HJB) equation of an informed buyer is

$$W(R_t) = \max\left[\sigma - V(R_t z_e), \frac{\lambda_t \mathbb{E}_t^i[W(R_t z_e^Q)] + \lambda_C(\sigma - V(R_t))}{\lambda_t + \lambda_C + \lambda_F}\right],$$

where the profit of immediate trading on exchange is $\sigma - V(R_t z_e)$ and the expected profit of trading in the dark pool is the weighted sum of:

- $\mathbb{E}_t^i[W(R_t z_e^Q)]$, the expected profit if the next event is the arrival of an exchange order.
- $\sigma - V(R_t)$, the profit if the next event is a dark pool cross.
- 0, the profit if the next event is the dividend payment.

To verify that $W(R_t) = \sigma - V(R_t z_e)$, it is sufficient to verify that, for all t and all realizations of random variable R_t,

$$\sigma - V(R_t z_e) > \frac{\lambda_t \mathbb{E}_t^i[\sigma - V(R_t z_e^Q z_e)] + \lambda_C(\sigma - V(R_t))}{\lambda_t + \lambda_C + \lambda_F}. \tag{89}$$

By Lemma 1, the expected profit for informed buyers to trade on the exchange is a supermartingale, that is, $\sigma - V(R_t z_e) > \mathbb{E}_t^i[\sigma - V(R_t z_e^Q z_e)]$. Thus, a sufficient condition for (89) is

$$\lambda_C + \lambda_F > \sup_{R \in (0,\infty)} \left\{\lambda_C \frac{\sigma - V(R)}{\sigma - V(R z_e)}\right\} = z_e \lambda_C,$$

which simplifies to (56).

Now we turn to a type-j liquidity buyer, whose HJB equation is

$$X(R_t) = \max\left[-(V(R_t z_e) - V(R_t)), \frac{\lambda_t \mathbb{E}_t[X(R_t z_e^Q)] - c_t^j}{\lambda_t + \lambda_C + \lambda_F}\right].$$

Her cost of liquidation $C(R_t)$ satisfies the HJB equation

$$C(R_t) = -X(R_t) = \min\left[V(R_t z_e) - V(R_t), \frac{\lambda_t \mathbb{E}_t[C(R_t z_e^Q)] + c_t^j}{\lambda_t + \lambda_C + \lambda_F}\right].$$

There are two cases, depending on j.

If $1 \le j < M$, to verify that $C(R_t) < V(R_t z_e) - V(R_t)$, it suffices to verify

$$V(R_t z_e) - V(R_t) > \frac{\lambda_t \mathbb{E}_t[V(R_t z_e^Q z_e) - V(R_t z_e^Q)] + c_t^j}{\lambda_t + \lambda_C + \lambda_F}, \tag{90}$$

where $C(R_t z_e^Q)$ is replaced by the higher cost of $V(R_t z_e^Q z_e) - V(R_t z_e^Q)$, as implied by the conjectured equilibrium. Because the ask spread is a supermartingale for liquidity buyers (Lemma 1), a sufficient condition for (90) is

$$\lambda_C + \lambda_F > \sup_{R\in(0,\infty)} \left\{ \frac{c_t^j}{V(Rz_e) - V(R)} \right\} = \sup_{R\in(0,\infty)} \left\{ \frac{\gamma_j \cdot 2R/(R+1)^2 \cdot \sigma}{V(Rz_e) - V(R)} \right\} = \gamma_j \frac{z_e}{z_e - 1},$$

which simplifies to (57).

If $M \le j \le J$, to verify that $C(R_t) = V(R_t z_e) - V(R_t)$, it suffices to verify

$$V(R_t z_e) - V(R_t) < \frac{\lambda_t \mathbb{E}_t[V(R_t z_e^Q z_e) - V(R_t z_e^Q)] + c_t^j}{\lambda_t + \lambda_C + \lambda_F},$$

that is, by Lemma 1,

$$(\lambda_C + \lambda_F)[V(R_t z_e) - V(R_t)] < -\lambda_t \frac{2R_t^2(z_e - 1)^3}{(R_t + 1)^2(R_t z_e + 1)(R_t z_e^2 + 1)} \sigma + \gamma_j \frac{2R_t}{(R_t + 1)^2} \sigma. \tag{91}$$

A sufficient condition for (91) is

$$\begin{aligned} \gamma_j &> \sup_{R\in(0,\infty)} \left\{ (\lambda_C + \lambda_F) \frac{(R+1)(z_e - 1)}{Rz_e + 1} \right\} + \sup_{R\in(0,\infty)} \left\{ \lambda_t \frac{R(z_e - 1)^3}{(Rz_e + 1)(Rz_e^2 + 1)} \right\} \\ &= (\lambda_C + \lambda_F)(z_e - 1) + \lambda_t \frac{(z_e - 1)^3}{z_e(\sqrt{z_e} + 1)^2}. \end{aligned}$$

The argument for sellers is symmetric and yields the same parameter conditions.

D List of Model Variables

This appendix summarizes key variables used in Section 3 and Section 4.

Variable	Description
	Variables Introduced in Section 3
v, σ	Asset value v is either $+\sigma$ or $-\sigma$, for $\sigma > 0$
$\bar{\mu}$, μ_I	Total masses of for-profit traders and informed traders
F	Cumulative distribution function (c.d.f.) of information-acquisition cost
Y	Signed informed trading interests: $Y = \text{sign}(v) \cdot \mu_I$
Z^+, Z^-, ϕ	Liquidity buy quantity Z^+ and liquidity sell quantity Z^- have p.d.f. ϕ
μ_z, σ_z^2	Total mean and variance of liquidity trading interests $Z^+ + Z^-$
c, γ, G	Delay cost of a liquidity trader is $c = \sigma\gamma$ per unit of asset, and γ has c.d.f. G
α_e, α_d, α_0	Fractions of liquidity traders who trade on the exchange, trade in the dark pool, and defer trading, respectively
β	Fraction of informed traders who trade in the dark pool
S	Exchange (effective) spread; bid is $-S$ and ask is S
$\bar{r}$	Dark pool crossing probability if no informed traders go to the dark pool
r^-, r^+	Dark pool crossing probabilities conditional on informed traders being on the same and opposite side, respectively
$\bar{\sigma}$	Maximum volatility for which informed traders avoid the dark pool
$\hat{\mu}_I(\sigma)$	Knife-edge mass of informed traders, defined by (62)
W_e, W_d	Expected profits of an informed buyer on the exchange and in the dark pool
$X_0(c)$, X_e, $X_d(c)$	Per-unit payoff of a liquidity buyer with a delay cost of c who defers trading, trades on the exchange, and trades in the dark pool, respectively
R_1	Period-1 log likelihood ratio of $\{v = +\sigma\}$ versus $\{v = -\sigma\}$
P_1	Period-1 closing price on the exchange
$I(\beta, \alpha_e)$	Signal-to-noise ratio of period-1 exchange order flow
V_b, V_s	Period-1 realized buy volume and sell volume on the exchange, respectively
V_d, V_e, V	Expected volumes in the dark pool, on the exchange, and both, respectively
	Variables Introduced in Section 4
x	Strictness of trade-at rule; maximum (minimum) dark pool price is xS ($-xS$)
y	Aggregate demand schedule in the dark pool if $Z^+ = 1$
y^+, y^-	Aggregate dark pool demand and supply schedules, respectively
p^*, H	Dark pool transaction price p^* has a c.d.f. of H
$X_d(p; c)$	Dark pool payoff of a liquidity buyer with the limit price p and per-unit delay cost c
$\bar{r}_x$	Dark pool crossing probability of a liquidity buyer with the limit price xS
$\bar{\sigma}(x)$	Maximum volatility for which informed traders avoid the dark pool

References

Admati, A. R., and Pfleiderer, P. 1988. A Theory of Intraday Patterns: Volume and Price Variability. *Review of Financial Studies* 1: 3–40.

Bennett, J., Colon, J., Feng, J., and Litwin, J. 2010. Ten Questions Every Institution Should Ask Their Dark Pool Providers. *Journal of Trading* 5: 10–13.

Bond, P., Edmans, A., and Goldstein, I. 2012. The Real Effects of Financial Markets. *Forthcoming, Annual Reviews of Financial Economics*.

Boni, L., Brown, D. C., and Leach, J. C. 2012. Dark Pool Exclusivity Matters. Working paper.

Boulatov, A., and George, T. J. 2010. Securities Trading when Liquidity Providers are Informed. Working paper University of Houston.

Brandes, Y., and Domowitz, I. 2010. Alternative Trading Systems in Europe: Trading Performance by European Venues Post-MiFID. *Journal of Trading* 5: 17–30.

Brunnermeier, M. K., and Pedersen, L. H. 2005. Predatory Trading. *Journal of Finance* 60: 1825–1863.

Buti, S., Rindi, B., Wen, Y., and Werner, I. M. 2011. Tick Size Regulation, Intermarket Competition and Sub-Penny Trading. Working paper.

Buti, S., Rindi, B., and Werner, I. M. 2011a. Diving into Dark Pools. Working paper Fisher College of Business, Ohio State University.

Buti, S., Rindi, B., and Werner, I. M. 2011b. Dynamic Dark Pool Trading Strategies in Limit Order Markets. Working paper.

Butler, G. 2007. Liquidity Aggregation: What Institutional Investors Need to Know. *Journal of Trading* 2: 108–113.

Carlin, B. I., Lobo, M. S., and Viswanathan, S. 2007. Episodic Liquidity Crisis: Cooperative and Predatory Trading. *Journal of Finance* 62: 2235–2274.

Carrie, C. 2008. Illuminating the New Dark Influence on Trading and U.S. Market Structure. *Journal of Trading* 3: 40–55.

CFA Institute 2009. Market Microstructure: The Impact of Fragmentation under the Markets in Financial Instruments Directive. Technical report CFA Institute.

Chordia, T., Roll, R., and Subrahmanyam, A. 2011. Recent Trends in Trading Activity and Market Quality. *Journal of Financial Economics* 101: 243–263.

Chowdhry, B., and Nanda, V. 1991. Multimarket Trading and Market Liquidity. *Review of Financial Studies* 4: 483–511.

Conrad, J., Johnson, K. M., and Wahal, S. 2003. Institutional trading and alternative trading systems. *Journal of Financial Economics* 70: 99–134.

CSA/IIROC 2009. Dark Pools, Dark Orders, and Other Development in Market Structure in Canada. Consulation paper 23-404 Canadian Securities Administration and Investment Industry Regulatory Organization of Canada.

Degryse, H., de Jong, F., and van Kervel, V. 2011. The Impact of Dark and Visible Fragmentation on Market Quality. Working paper.

Degryse, H., Van Achter, M., and Wuyts, G. 2009. Dynamic Order Submission Strategies with Competition between a Dealer Market and a Crossing Network. *Journal of Financial Economics* 91: 319–338.

Domowitz, I., Finkelshteyn, I., and Yegerman, H. 2009. Cul de Sacs and Highways, An Optical Tour of Dark Pool Trading Performance. *Journal of Trading* 4: 16–22.

Easley, D., Keifer, N. M., and O'Hara, M. 1996. Cream-Skimming or Profit-Sharing? The Curious Role of Purchased Order Flow. *Journal of Finance* 51: 811–833.

European Commission 2010. Review of the Markets in Financial Instruments Directive (MiFID). Public consultation European Commission.

Foster, D., Gervais, S., and Ramaswamy, K. 2007. The Benefits of Volume-Conditional Order-Crossing. Working paper.

Glosten, L. R., and Milgrom, P. R. 1985. Bid, Ask and Transaction Prices in a Specialist Market with Heterogenously Informed Traders. *Journal of Financial Economics* 14: 71–100.

Hendershott, T., and Jones, C. M. 2005. Island Goes Dark: Transparency, Fragmentation, and Regulation. *Review of Financial Studies* 18: 743–793.

Hendershott, T., and Mendelson, H. 2000. Crossing Networks and Dealer Marekts: Competition and Performance. *Journal of Finance* 55: 2071–2115.

International Organization of Securities Commissions 2010. Issues Raised by Dark Liquidity. Consultation report Technical Committee of the International Organization of Securities Commissions.

International Organization of Securities Commissions 2011. Principles for Dark Liquidity. Final report Technical Committee of the International Organization of Securities Commissions.

Jiang, C., McInish, T., and Upson, J. 2011. Why Fragmented Markets Have Better Market Quality: The Flight of Liquidity Order Flows to Off Exchange Venues. Working paper.

Johnson, B. 2010. *Algorithmic Trading & DMA: An introduction to direct access trading strategies*: . 4Myeloma Press, London.

Joint CFTC-SEC Advisory Committee 2011. Recommendations regarding Regulatory Responses to the Market Event of May 6, 2010. Summary report.

Kyle, A. S. 1985. Continuous Auctions and Insider Trading. *Econometrica* 53: 1315–1335.

MacAfee, R. P. 1992. A Dominant Strategy Double Auction. *Journal of Economic Theory* 56: 434–450.

Mittal, H. 2008. Are You Playing in a Toxic Dark Pool? A Guide to Preventing Information Leakage. *The Journal of Trading* 3: 20–33.

Næs, R., and Odegaard, B. A. 2006. Equity trading by institutional investors: To cross or not to cross?. *Journal of Financial Markets* 9: 79–99.

Nimalendran, M., and Ray, S. 2012. Informational Linkages Between Dark and Lit Trading Venues. Working paper University of Florida.

O'Hara, M., and Ye, M. 2011. Is Market Fragmentation Harming Market Quality?. *Journal of Financial Economics* 100: 459–474.

Pagano, M. 1989. Trading Volume and Asset Liquidity. *Quarterly Journal of Economics* 104: 255–274.

Parlour, C. A., and Seppi, D. J. 2008. Limit Order Markets: A Survey. *Chapter 3, Handbook of Financial Intermediation and Banking.*

Ready, M. J. 2012. Determinants of Volume in Dark Pools. Working paper University of Wisconsin-Madison.

Rosenblatt Securities 2011. Let there be light. Market structure analysis.

Saraiya, N., and Mittal, H. 2009. Understanding and Avoiding Adverse Selection in Dark Pools. Working paper Investment Technology Group.

Securities and Exchange Commission 2005. Regulation NMS. *Federal Register* 70: 37496–37644.

Securities and Exchange Commission 2009. Regulation of Non-Public Trading Interests; Proposed Rule. *Federal Register* 74: 61207–61238.

Securities and Exchange Commission 2010. Concept Release on Equity Market Structure; Proposed Rule. *Federal Register* 75: 3593–3614.

Sofianos, G., and Xiang, J. 2011. Dark Pool Races, Part Two. Working paper. Goldman Sachs.

Sun, Y. 2006. The exact law of large numbers via Fubini extension and characterization of insurable risks. *Journal of Economic Theory* 126: 31–69.

Tabb Group 2011. Liquidity Matrix. Technical report.

Weaver, D. 2011. Off-Exchange Reporting and Market Quality in a Fragmented Market Structure. Working paper.

Ye, M. 2010. Non-execution and Market Share of Crossing Networks. Working paper University of Illinois.

Ye, M. 2011. A Glimpse into the Dark: Price Formation, Transaction Costs and Market Share of the Crossing Network. Working paper University of Illinois.